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Alain Sellier, Alain Millard. A homogenized formulation to account for sliding of non-meshed reinforcements during the cracking of brittle matrix composites: Application to reinforced concrete. *Engineering Fracture Mechanics*, 2019, 213, pp.182-196. 10.1016/j.engfracmech.2019.04.008 . hal-02099138

HAL Id: hal-02099138

<https://hal.insa-toulouse.fr/hal-02099138>

Submitted on 14 Apr 2019

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A homogenized formulation to account for sliding of non-meshed reinforcements during the cracking of brittle matrix composites : application to reinforced concrete

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Abstract

Non-linear finite element modelling of complex structures made of composites, such as reinforced concrete, remains a challenge because, until now, the only way to consider the important phenomenon of sliding between the reinforcements and the brittle matrix of the composite has been to mesh the reinforcements and their interfaces explicitly . This method is accurate but so expensive in terms of computational resources that only critical small elements of composites structures are modelled using it. To get around this limit, a method avoiding the meshing of composite reinforcements is proposed. It consists in treating the sliding between reinforcements and matrix with a differential formulation that provides the deformation of reinforcements directly as a continuous field superimposed to the displacement field of the matrix. The method needs a minor modification of the finite element code, which can take advantage of its analogy with the anisotropic thermal formulation. After the analytical presentation of the method, two theoretical cases of study are given to confront the results obtained with this method without meshing of reinforcements, with reference results obtained using a complete mesh of the matrix, reinforcements and interfaces.

Keywords: Concrete, Damage, Reinforcements, Cracking, Finite elements

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1. Introduction

Context. In civil engineering applications, crack opening in reinforced elements is a limit state to be controlled [44], because cracks are privileged ways for the ingress of deleterious agents. For instance, water and carbonic gas ingress rapidly in cracks and cause corrosion of the reinforcements. In other structures, such as water tanks and nuclear containment vessels, cracks opening is forbidden in normal conditions of exploitation, and when cracks occur in accidental conditions, the leakage flow must be limited to avoid dissemination of dangerous elements into the environment. So, predicting the crack opening and permeability of such structures is an objective for engineers dealing with these problems. Although progress has been made in recent decades to link crack opening and leakage flow [12, 31, 32, 30], the problem of crack opening assessment is still a major concern [36, 24].

Problem to solve. This problem is difficult to solve for two main reasons: the concrete has weak and relatively random tensile strength [35, 3], and the reinforcements slide relatively to the concrete matrix during the formation of cracks [2, 13, 25]. The randomness of concrete tensile strength can be treated using various methods that are usable at different scales of modelling [43, 39, 4] but the sliding between reinforcements and the matrix can be treated only at the scale of the reinforcements, meshing them and their interfaces with the matrix explicitly [13, 6, 17, 22, 23] in order to consider the behavior law of the interface in the structural model. The consequence, in terms of computational resources, is problematic because, in the context of finite element modelling, the mesh size becomes controlled by the size of, and the spacing between, reinforcements, which leads to a number of nodes proportional to the size of the structure and prevents the use of large elements for large structures. For instance, the most complex numerical models currently used for a nuclear power plant containment vessel of more than 30 m diameter need to mesh all the reinforcements and pre-stressed wires. The spacing between reinforcements and wires being only a few decimeters, the finite element dimensions should be constrained to this size and, consequently, the number of finite elements will be far too high for engineering applications. So, to simplify the problem, the mesh is generally composed of larger finite elements, and kinematic relations between nodes of massive finite elements and nodes of segments used to mesh reinforcements are used. These kinematic relations assume a perfect bond between the reinforcements

37 and the concrete, so, even if all the reinforcements are meshed [1, 5], the
 38 crack prediction is still not accurate because possible sliding between the
 39 two components is neglected. Until a method is found to consider the inter-
 40 action between reinforcements and matrix in a very simple way, it will be
 41 difficult to improve the realism of models. That is our reason for proposing
 42 the present method.

43 *Principle of the proposed method.* This method is able to consider the slid-
 44 ing between reinforcements and the brittle matrix without meshing the rein-
 45 forcements and their interfaces. It is based on the classic principle that the
 46 reinforced matrix can be modelled at large scales by a homogenized behavior
 47 law mixing the contributions of matrix and reinforcements. However, unlike
 48 classic homogenization methods, which consider reinforcements as inclusions
 49 in a representative elementary volume, the present method takes advantage
 50 of the finite element context to use a non-local formulation to assess rein-
 51 forcements deformations, taking not only the sliding within but also outside
 52 the representative elementary volume into account. That is its main speci-
 53 ficity. Finally, the strains in reinforcements are modelled using a continuous
 54 field that does not need the reinforcements to be meshed. Only their local
 55 volumetric fractions and their orientations are needed. These can be sup-
 56 plied to the finite element code as material parameter fields, independently
 57 of the underlying mesh. The paper first presents the theory of this method,
 58 then two virtual applications allow the method solution (coarse mesh with-
 59 out meshing of reinforcements) to be confronted with a reference solution
 60 obtained with a fine mesh including reinforcements and their interfaces.

61 **2. Theoretical background**

62 *2.1. Equilibrium equation of a reinforcement*

63 The local equilibrium of a cylindrical reinforcement section illustrated in
 64 Figure 1, along the local x axis, can be written:

$$\frac{\partial \sigma^r}{\partial x} \frac{\pi (D^r)^2}{4} + \tau^{m/r} \pi D^r = 0 \quad (1)$$

65 with σ^r the axial stress in the reinforcement, D^r its diameter and $\tau^{m/r}$ the
 66 shear stress applied by the matrix on the reinforcement along the interface.

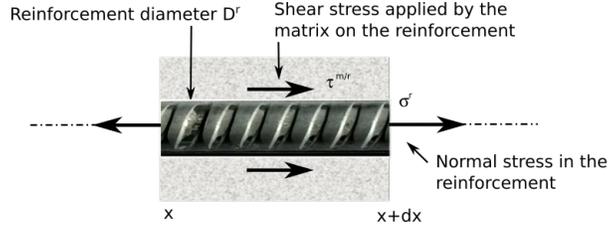


Figure 1: Axial and shear stresses applied to a reinforcement imbedded in a matrix

67 *2.2. Behavior law of the reinforcement*

68 The stress in the reinforcement is assumed to be coaxial with x , so its
69 behavior law can be summed up in (2).

$$\sigma^r = E^r \underbrace{(\epsilon^r - \epsilon^{ra})}_{\epsilon^{re}} \quad (2)$$

70 with E^r the Young's modulus of the reinforcement, ϵ^{re} its elastic strain, ϵ^r its
71 axial strain and ϵ^{ra} its an-elastic axial strain including plastic, visco-plastic
72 [10] and thermal strain.

73 *2.3. Behavior law of the interface*

74 In (1), the shear stress $\tau^{m/r}$ along the interface is assumed to depend
75 only on the relative axial displacement $g^{m/r}$ between the matrix and the
76 reinforcement (3).

$$\tau^{m/r} = K^i (g^{m/r} - g^{m/r a}) \quad (3)$$

77 with K^i the stiffness of the interface, $g^{m/r}$ the relative axial displacement
78 between matrix and reinforcement, and $g^{m/r a}$ the an-elastic relative dis-
79 placement. The behavior law of interface (3) is usually identified with a
80 "pull-out" test [15] such as that illustrated in Figure 2.3. This figure is an il-
81 lustration of a typical pull-out test obtained with a notched bar. Practically,
82 the shape of the curve can be modified according to the material character-
83 istics, bar diameter or notch height. The behavior law can also be expressed
84 incrementally using the tangent stiffness H^i :

$$d\tau^{m/r} = H^i dg^{m/r} \quad (4)$$

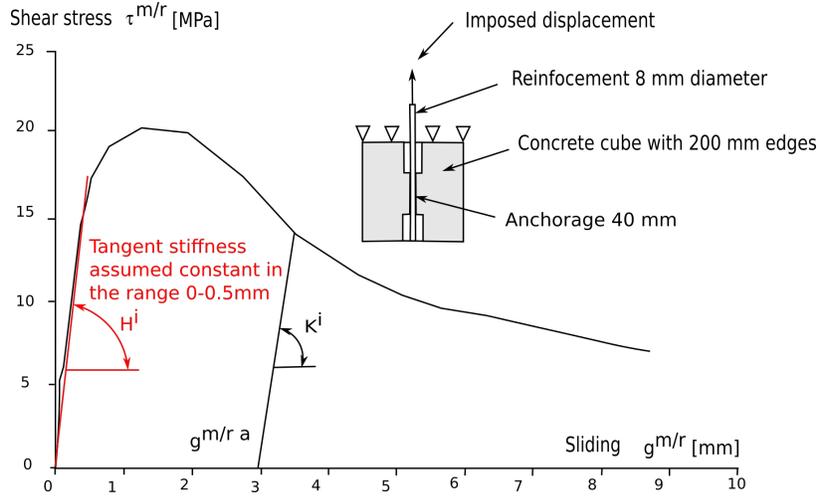


Figure 2: Interface behavior law identified with a pull-out test performed by N.Handika on steel rebar diameter 8 mm with lugs imbedded of 40 mm in a concrete block of 200 mm edges. The concrete characteristic were $R_c = 56\text{MPa}$, $R_t = 3.9\text{MPa}$, $E = 38500\text{MPa}$, [18]

85 2.4. Application domain

86 For the case of steel bars for reinforced concrete with lugs, according to
87 experimental results shown in Figure 2.3, the initial tangent stiffness H^i can
88 be kept constant until sliding reaches around $500\ \mu\text{m}$. This approximation
89 can be exploited to simplify the numerical implementation of the non-local
90 behavior law as explained below, but can also be avoided using an updating
91 process of H^i in the numerical model. However, for the sake of simplicity,
92 the non-local formulation is clarified below with the assumption of a constant
93 tangent stiffness, H^i , that limits its current application domain to the sliding
94 range $[0 - 500\ \mu\text{m}]$ in case of application to reinforced concrete. It is worth
95 noting that a sliding of $[0 - 500\ \mu\text{m}]$ corresponds to the half crack opening
96 (cf. Figure 3), the maximal crack opening conceivable with the simplified
97 formulation is then $1\ \text{mm}$. This is sufficient for most reinforced concrete
98 applications because the serviceability limit state is usually below $300\ \mu\text{m}$
99 [11], and the ultimate state corresponds to the reinforcement plasticity, which
100 generally occurs under $1\ \text{mm}$ of crack opening. For other materials, such as
101 carbon fiber composites or fiber concrete, the validity domain of this approx-
102 imation will have to be defined before any application.

103 *2.5. Kinematic equation*

104 In a multi-cracked matrix, the sliding is maximal at the crack location
 105 and decreases with the distance from the crack until the symmetry plane as
 106 shown schematically in Figure 3. So, the sliding at the location of the crack
 107 can then be computed as the integral of difference in axial strains between
 108 reinforcement and matrix (5) from a symmetry plane between two cracks
 109 ($x = 0$).

$$g^{m/r}(x) = \int_{\xi=0}^x (\epsilon^m - \epsilon^r) d\xi \quad (5)$$

110 with $x = 0$ at the symmetry plane in Figure 3, and $x = x^c$ at the crack
 111 location relative to the symmetry plane.

112 *2.6. Resulting differential formulation*

113 In order to obtain a simple formulation combining the equilibrium equa-
 114 tion (1), the behavior equations of the reinforcement (2) and of the interface
 115 (4), and the kinematic relation of sliding (5) in its derivative form, the equi-
 116 librium equation can be derived with respect to x and combined with the
 117 differential formulations of the behavior laws (6).

$$\left\{ \begin{array}{l} \frac{\partial^2 \sigma^r}{\partial x^2} \frac{D^r}{4} + \frac{\partial \tau^{m/r}}{\partial x} = 0 \\ \frac{\partial x^2}{\partial \tau^{m/r}} = E^r \frac{\partial x^2}{\partial \epsilon^{re}} \\ \frac{\partial x}{\partial g^{m/r}} = H^i \frac{\partial x}{\partial g^{m/r}} \\ \frac{\partial g^{m/r}}{\partial x} = \epsilon^m - (\epsilon^{re} + \epsilon^{ra}) \end{array} \right. \quad (6)$$

118 Once combined, the set of equations (6) leads to the resulting form (7).

$$\epsilon^{re} - \frac{E^r D^r}{4H^i} \frac{\partial^2 \epsilon^{re}}{\partial x^2} = \epsilon^m - \epsilon^{ra} \quad (7)$$

119 In (7), the elastic strain in the reinforcement (ϵ^{re}) appears to be the result
 120 of a second order differential equation in space analogous to the classical
 121 Helmholtz equation form (8). This type of equation is sometimes also used
 122 in mechanics to regularize finite element problems for which the material be-
 123 havior law presents a softening leading to a crack localization [28, 26]. In this
 124 case, the Helmholtz form, also known as "second gradient formulation" or

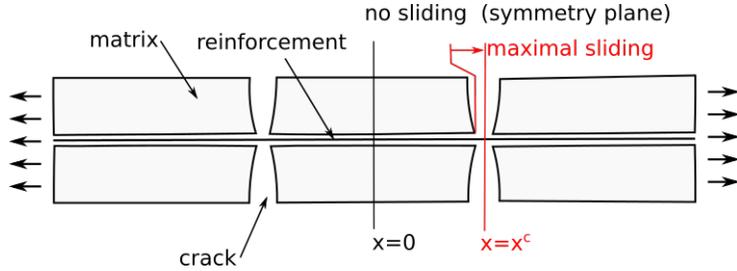


Figure 3: Crack periodicity, symmetry plane and maximal sliding at the crack tips

125 "phase field formulation" of the "non local theory", allows the internal vari-
 126 ables controlling the softening to be spread over a zone that is independent
 127 of the finite element sizes. Another application of the Helmholtz equation
 128 is proposed in [39] to consider the Weibull scale effect in a simplified way.
 129 Equation (7) then constitutes the third application of this type of equation
 130 in solid mechanics.

$$\epsilon^{re} - \frac{l_c^2}{2} \frac{\partial^2 \epsilon^{re}}{\partial x^2} = S \quad (8)$$

131 In (8), l_c^r is a characteristic diffusion length and S is a source term. The
 132 analogy between (7) and (8) leads to the identification of these terms. The
 133 characteristic length l_c^r is given by (9), and the source term by (10).

$$l_c^r = \sqrt{\frac{E^r D^r}{2H^i}} \quad (9)$$

134

$$S = \epsilon^m - \epsilon^{ra} \quad (10)$$

135 With this formulation the elastic strain in the reinforcement appears analog-
 136 ous to the diffusion of the term $(\epsilon^m - \epsilon^{ra})$. It is worth noting that as long as
 137 $\epsilon^{ra} = 0$, the over-tension in the reinforcement due to the sliding is analogous
 138 to the diffusion of the strain ϵ^m in the finite element where the crack occurs.
 139 In other words, the sliding displacement along the rebar can be seen as a
 140 "diffusion" of the crack displacement jump over a length controlled by l_c^r .

141 2.7. Boundary conditions

142 Concerning the boundary conditions, if there is no sliding on the edges
 143 $\partial\Omega$ of the integration domain Ω (perfect anchorage at the edges), a Neumann

144 condition can be used for the state variable ϵ^{re} (11).

$$\frac{\partial \epsilon^{re}}{\partial x} = 0 \text{ if } x \in \partial\Omega \quad (11)$$

145 In fact putting this condition into (8) leads to (12).

$$\epsilon^r = \epsilon^m \text{ if } x \in \partial\Omega \quad (12)$$

146 If (12) is true, it means that strains are the same in the matrix and the
147 rebar, and there is then no sliding. Other types of boundary conditions may
148 be used. For instance, to simulate a pull out, a Dirichlet condition could be
149 used for ϵ^{re} , but, for the sake of simplicity, the applications below use only
150 condition (12). In fact, the null Neumann boundary condition is the default
151 condition of any formulation in the finite element codes, so this condition
152 does not need to be specified in the code. This condition is realistic for
153 some problems where sliding does not occurs perpendicularly to the edges.
154 Specifically if the edges are free of stresses or weakly loaded, or subjected to
155 imposed displacements.

156 **3. Implementation in a finite element code to model reinforced** 157 **brittle matrix**

158 The interest of the differential form of the sliding reinforcement problem
159 lies in its ability to be used in homogenized behavior laws of composites.
160 Instead of meshing all the reinforcements, the interfaces and the matrix, the
161 reinforcement elastic strain is treated as a diffuse field superimposed on the
162 displacement field. To take advantage of this method, the finite elements code
163 must be modified to be able to treat the two fields simultaneously. On the
164 one hand, the equilibrium of the homogenized material has to be considered,
165 and, on the other hand, the Helmholtz formulation provides the elastic strain
166 of reinforcements. Once the two fields are known, the displacement field
167 provides the matrix strains ϵ^m and the stresses in the matrix, while the field
168 of ϵ^{re} enables the stresses field in the reinforcements to be computed.

169 *3.1. Equations to be solved*

170 The main balance equations to be solved are summarized below. The
171 main state variables are the displacement \vec{U} and the elastic strains in rein-
172 forcements ϵ_n^{re} , with 'n the reinforcement number. They are solved by the

173 balance equations. Next, the state laws (behavior laws and evolution laws to
 174 account for non-linearities) can be used to assess the internal variables and
 175 stresses.

176 3.1.1. Balance equations

177 Two types of equilibrium equations have to be solved simultaneously:

- 178 • The classical stresses balance at the scale of the homogenized material
 179 (13). At this scale, the stresses are σ_{ij} , with i, j the subscripts corre-
 180 sponding to the global system coordinates and f_i the volume force in
 181 direction x_i .
- 182 • The shear stress balance at each interface between the matrix and a
 183 given reinforcement considered by (8).

184 These two sets of equations are summarized in (13).

$$\begin{cases} \sum_{j=1}^3 \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0 \text{ for } i \in [1, 2, 3] \\ \epsilon_n^{er} - \frac{l_{c_n}^2}{2} \frac{\partial^2 \epsilon_n^{er}}{\partial x_n^2} = S_n \text{ for } n \in [1..N^r] \end{cases} \quad (13)$$

185 In (13) N^r is the number of reinforcement types considered in the homoge-
 186 nized behavior law clarified below, x_n is the local coordinate along the rein-
 187 forcement number n , and ϵ_n^{er} is the axial elastic deformation of a reinforce-
 188 ment.

189 3.1.2. State laws for each phase of composite

190 The state laws include the behavior law of the matrix, the behavior law of
 191 the reinforcements and the method for combining the stresses deduced from
 192 these two. A brief presentation of these three aspects of the homogenized
 193 behavior law is given below for reinforced concrete.

194 *Homogenized stresses in the composite.* The homogenized behavior law can
 195 be obtained using different homogenization methods, but, for the sake of
 196 simplicity, the simplest combination is used in the following. The homoge-
 197 nized stress σ_{ij} is simply obtained by summing of the matrix contribution
 198 and reinforcements contributions (14).

$$\sigma_{ij} = (1 - \sum_{n=1}^{N^r} \rho^n) \sigma_{ij}^m + \sum_{n=1}^{N^r} \rho^n \sigma_{ij}^{rn} \quad (14)$$

199 In (14), ρ^n is the volumetric fraction of reinforcement number n , σ_{ij}^m the stress
 200 in the matrix and σ_{ij}^{rn} the tensor component obtained with the reinforcement
 201 stresses (2) multiplied by the orientation tensors \bar{P}^{rn} (15).

$$P_{ij}^{rn} = e_i^{rn} e_j^{rn} \quad (15)$$

202 In (15), e_i^{rn} is a component of the unit vector $e^{\bar{r}n}$ giving the orientation
 203 of the reinforcement in the matrix. Equation (15) does not consider
 204 the reorientation of the force in the reinforcement due to the dowel effect
 205 occurring when a crack opens in a direction different than the reinforcements
 206 ones. This dowel effect could be added using a method to compute the real
 207 directions of the forces in the reinforcements crossing the cracks.[38]

208 *Stresses in the matrix.* For the matrix (stresses σ_{ij}^m in (14)), the behavior
 209 law is derived from a model already described in [37]. It is a law based on
 210 plasticity and anisotropic damage. This law allows the softening behavior of
 211 the matrix to be considered. The fracture energy is managed using a local
 212 method derived from the Hillerborg principle [19]: in each principal direction
 213 of stresses in the matrix \vec{e}_I , the dimension l_I of the finite element is assessed
 214 using coordinates of the finite element nodes and their interpolation functions
 215 [41], and the softening branch of the behavior law is automatically adjusted
 216 to ensure the energy dissipated will be equal to the imposed fracture energy
 217 G_f . As the model is anisotropic, the principal stresses are assessed in the
 218 principal directions of effective stresses ($\tilde{\sigma}$) (16).

$$\sigma_I^m = (1 - D^c)(\tilde{\sigma}_I^{m-} C_I^c + (1 - D_I^t)\tilde{\sigma}_I^{m+}) \quad (16)$$

219 In (16), $\tilde{\sigma}_I^{m-}$ stands for the negative principal effective stresses and $\tilde{\sigma}_I^{m+}$
 220 for the positive principal effective stresses. The damage D^c stands for the
 221 micro-cracking effect on the concrete stiffness: $D^c \rightarrow 1$ if the matrix is totally
 222 crushed and $D^c = 0$ for the undamaged matrix. This damage is driven by the
 223 plastic strains ϵ_{ij}^{mpc} induced by the yielding of a Drucker Prager criterion [14].
 224 In (16) D_I^t is the tensile localized damage in the principal direction I . This
 225 damage depends on the maximum values reached by principal values of the
 226 plastic strains ϵ_{ij}^{mpt} induced by the yielding of principal stress criteria in each
 227 principal direction of $\tilde{\sigma}^m$. The decomposition of the effective stress tensor
 228 or strain tensor into positive and negative parts is a classic way to properly
 229 consider the two types of cracking possible in concrete but other decompo-
 230 sitions could be used [34, 9]. C_I^c is a crack re-closure function, which also

231 depends on the plastic strain ϵ_{ij}^{mpt} . This function allows us, for an existing
 232 localized crack already re-closed, to consider that if the crack re-opens, the
 233 contacts disappear between its edges ($C_I^c \rightarrow 0$), while the contacts reappear
 234 progressively when the crack re-closes ($C_I^c \rightarrow 1$). This is due to the roughness
 235 of the crack faces, which induce a progressive recovery of stiffness under neg-
 236 ative stresses [21]. The crack re-closure is controlled with a principal stress
 237 criterion while the corresponding principal plastic strain ϵ_I^{mpt} stays positive,
 238 but, when this strain becomes zero, the criterion is deactivated in this di-
 239 rection and only the Drucker-Prager criterion controls the negative principal
 240 stresses. The Drucker-Prager criterion is a shear criterion sensitive to hydro-
 241 static pressure: it considers the effect of the tri-axiality of the stress state on
 242 the compressive strength. An example of a cyclic test including damage in
 243 tension, damage and plasticity in compression, and tensile crack re-closures is
 244 given in Figure 4. In this model, used to manage the cracking of the matrix,
 245 the crack opening (w_I) is included in the finite element displacement field
 246 [30]. This feature allows ϵ^m to be used directly in (10) as explained above .
 247 If the interpolation functions used in the finite element for the displacement
 248 are linear, the relationship between the crack opening and the strain in the
 249 finite element is approximated using the plastic strain ϵ_I^{mpt} (17).

$$w_I = \epsilon_I^{mpt} l_I \quad (17)$$

250 The link between the crack opening and the tensile damage is given by equa-
 251 tion (18).

$$D_I^t = 1 - \left(\frac{w_I^k}{w_I^k + \max(w_I)} \right)^2 \quad (18)$$

252 In (18), w_I^k is a parameter linked to the fracture energy G_f . The link between
 253 w_I^k and the fracture energy depends on the finite element size l_I in the
 254 principal direction of tension. The relationship between w_I^k , the fracture
 255 energy G_f , and the finite element length l_I , is given by equation (19).

$$G_f = l_I R^t \left(\frac{R^t}{2E^m} + w_I^k \right) \quad (19)$$

256 In (19), E^m is the Young's modulus of the matrix and R^t its tensile strength.
 257 This relationship is the consequence of the Hillerborg principle: the energy
 258 consumed by a crack propagation is surfacic [19], while the energy computed
 259 by the program is proportional to the volume of the damaged finite element,

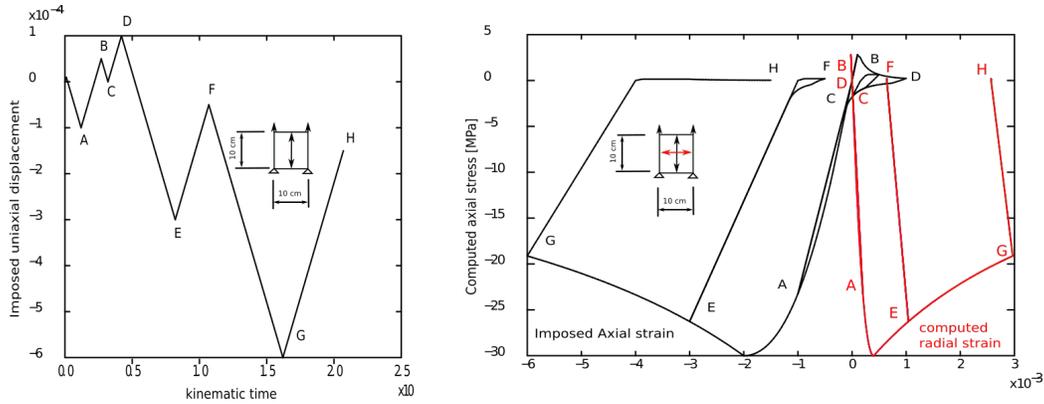


Figure 4: Matrix model response to a uniaxial cyclic test, for $E^m = 30GPa$, $R_c^m = 30MPa$, $R_t^m = 3MPa$, $G_f = 100J/m^2$, the finite element is a single cube having 10 cm edges.

260 so the volumetric energy has to be adapted (19) to satisfy the condition of
 261 equation (20).

$$G_f = l_I \int_0^\infty \sigma_I^m d\epsilon_I^{mpt} \quad (20)$$

262 Equation (20) shows that the element length (l_I) has to be assessed in the
 263 principal direction of the stresses in the matrix.

264 *Stresses in the reinforcements.* The behavior law for the reinforcements is
 265 elasto-plastic, with a kinematic linear hardening. The uniaxial behavior law
 266 is given by equation (2). It is also possible to take account of visco-plastic
 267 strain as explained in [10]. This feature is needed when dealing with pre-
 268 stressed wires, for example. For the sake of simplicity, this is not the case in
 269 the following applications.

270 3.2. Finite Element Formulation

271 As explained in the introduction, the objective is to avoid meshing the re-
 272 inforcements with bars, interfaces, and so on. So the massive finite elements,
 273 used for the homogenized material, support both the displacement field and
 274 the elastic strain field of "distributed" reinforcements. For instance, in 3 di-
 275 mensions, each node of the finite element model supports the state variables

276 vector (21).

$$\begin{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \\ \begin{bmatrix} \epsilon_1^{er} \\ \dots \\ \epsilon_n^{er} \end{bmatrix} \end{bmatrix} \quad (21)$$

277 In (21), the first three variables are the displacements solved by the bal-
 278 ance equations applied to the homogenized material, the last ones are the
 279 elastic strains of reinforcements, each one being solved by a Helmholtz equa-
 280 tion corresponding to the condition of local equilibrium between matrix and
 281 reinforcements as explained above.

282 3.2.1. Variational form of Helmholtz equation

283 For a reinforcement oriented in a given direction x_n , the variational form
 284 of the Helmholtz equation (8) is written (22).

$$\int_{\Omega} \psi_n \epsilon_n^{er} dx_n - \int_{\Omega} \psi_n \frac{l_{c_n}^2}{2} \frac{\partial^2 \epsilon_n^{er}}{\partial x_n^2} dx_n - \int_{\Omega} \psi_n S_n dx_n = 0 \quad \forall \psi_n \quad (22)$$

285 In (22), ψ_n is the test function. Using an integral transformation, this equa-
 286 tion leads to the second variational form (23).

$$\int_{\Omega} \psi_n \epsilon_n^{er} dx_n - \left[\psi_n \frac{l_{c_n}^2}{2} \frac{\partial \epsilon_n^{er}}{\partial x_n} \right]_{\partial \Omega} + \int_{\Omega} \frac{\partial \psi_n}{\partial x_n} \frac{l_{c_n}^2}{2} \frac{\partial \epsilon_n^{er}}{\partial x_n} dx_n = \int_{\Omega} \psi_n S_n dx_n \quad \forall \psi_n \quad (23)$$

287 with $\partial \Omega$ the edges of the meshed domain Ω .

288 3.2.2. Finite Element formulation

289 Taking the boundary conditions (11) into account, once discretized on the
 290 mesh and integrated over the whole structure by taking advantage of the finite
 291 element interpolation functions, the second form (23) becomes equivalent to
 292 the linear problem (24).

$$\underbrace{\left[\bar{C} + \bar{K}_n \right]}_{\bar{K}_n^x} \bar{\epsilon}_n^{er} = \bar{S}_n^r \quad \forall n \quad (24)$$

293 In (24) \bar{C} is a capacity matrix obtained by assuming a homogeneous unit
 294 capacity in the material, \bar{K}_n is an anisotropic conductivity matrix deduced
 295 from the equivalent material conductivity (25), and \bar{S}_n is the source term re-
 296 assessed at each step of loading with (10). \bar{K}_n^r is the linear system resulting
 297 assembly of \bar{C} and \bar{K}_n .

$$\bar{K}_n = \frac{l_{cn}^2}{2} e_n^r \otimes e_n^r \quad (25)$$

298 In (25) e_n^r is the local orientation of reinforcement number n . As the source
 299 term must be updated for each step of loading, the solving of (24) can be
 300 inserted in the global loop of non-linear-resolution of the finite element soft-
 301 ware: First the resolution of the equilibrium supplies the displacement incre-
 302 ment field $(\Delta u, \Delta v, \Delta w)$, which is used to compute the strain increment in
 303 the matrix $\Delta \epsilon^m$, and the anelastic strain in the reinforcement ϵ_n^{ar} is initialized
 304 with the solution of the last converged step. These two terms are used to
 305 update the source term of (24). Once (24) is solved, the new elastic strain in
 306 the reinforcement (ϵ_n^{er}) is known and can be used to compute the stress in the
 307 reinforcement using (2). If plastic yielding occurs in the reinforcement, the
 308 source term is updated until convergence. Otherwise, the stress is directly
 309 used to compute the homogenized response of the material using equation
 310 (14). Finally at each sub-step of the non-linear procedure, the linear system
 311 to be solved is summarized in (26).

$$\begin{cases} \bar{K} \Delta \bar{U} = \Delta \bar{F} \\ \Delta \bar{S}_n^r = \text{sym}(\nabla(\Delta \bar{U})) - \Delta \epsilon_n^{\bar{r}a} \quad \forall n \\ \bar{K}_n^r \Delta \epsilon_n^{\bar{e}r} = \Delta \bar{S}_n^r \quad \forall n \end{cases} \quad (26)$$

312 with \bar{K} the stiffness matrix of the structure, $\Delta \bar{U}$ the nodal increments of
 313 displacement, $\Delta \bar{F}$ the applied forces to be balanced (real forces increment
 314 for the first sub-step, or unbalanced internal forces during the iterative solv-
 315 ing), $\text{sym}(\nabla(\Delta \bar{U}))$ the symmetric part of the displacement increment gra-
 316 dient, and $\Delta \epsilon_n^{\bar{r}a}$ the an-elastic strain increment projected from the Gauss
 317 points to the nodes of the finite elements. This resolution method has been
 318 implemented in the finite element code Cast3m [7], in two steps at each it-
 319 eration: first the equilibrium is solved to assess the displacement increments
 320 and strains, then the source term of the Helmholtz equation is deduced from
 321 the deformations and used to assess the elastic strain in the reinforcements.
 322 Once known, the stresses in the reinforcements and matrix are combined and

323 the global equilibrium is tested. The procedure is iterated until the global
324 equilibrium is verified.

325 4. Applications

326 First, two theoretical cases are treated in order to show the aptitude of
327 the proposed method to consider correctly the linear de-bonding of a rein-
328 forcement in a simple reinforced concrete tie. Secondly a case of study is
329 provided to illustrate the applicability of the method to a real structure with
330 a more complex reinforcement system.

331 4.1. Theoretical cases

332 The objective of this section is to provide two elementary applications
333 intended to test the numerical implementation of the method. First a simple
334 reinforced concrete tie beam with a single crack is analyzed, then a second,
335 longer tie with three cracks is studied.

336 4.1.1. Theoretical case with a single crack in a reinforced concrete tie beam

337 The first application concerns a theoretical reinforced concrete tie beam
338 for which the homogenized finite element solution obtained with mesh (b) in
339 Figure 5, is compared to a reference finite element solution obtained with the
340 detailed mesh (a) in Figure 5. As can be observed in the Figure, the mesh (b)
341 is simpler than (a) and number of nodes is considerably reduced. Even if each
342 node of (b) supports the full state variables vector (21) instead of only the
343 displacements, the computational duration is divided by 5, especially because
344 the number of Gauss points where the behavior laws are integrated is reduced.
345 And, in non-linear numerical models, more computational time is consumed
346 for local non-linear behavior law solving (14) than for the linear resolution
347 of the global system (24). The material characteristics of the reinforced
348 concrete tie beam are given in Table 1. The tensile strength given in table 1
349 corresponds to the weakest zone. In the other zones of the tie the strength is
350 three times higher to avoid any damage out of the predefined weakest zone.
351 The interface stiffness given in table 1 is $40GPa$, it corresponds to a secant
352 modulus of $300GPa$ in an interface zone $3mm$ thick with a Poisson coefficient
353 $\nu = 0.25$. The reference solution is given in figure 6. It is Worth noting
354 that this reference solution is based on two important assumptions: first
355 the interface behavior is linear, secondly the concrete cracking is controlled
356 using a Hillerborh method which considers the crack included in the finite

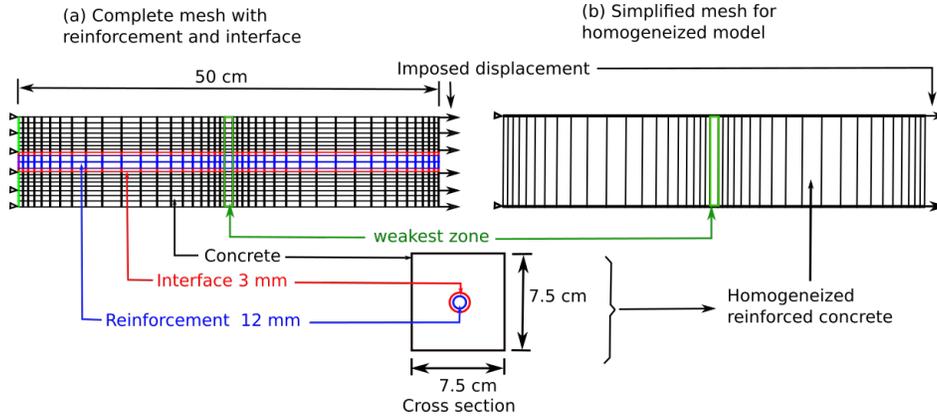


Figure 5: Complete mesh (a) and simplified mesh (b) for the single crack test

357 element, so that the strain in the finite element where the crack takes place,
 358 multiplied by the finite element length in the direction of crack opening,
 359 gives the crack opening, which is independent of the mesh size thanks to
 360 the Hillerborgh method. Due to this method, the finite element size where
 361 the crack takes place can be chosen freely. An interesting choice consists to
 362 adopt a dimension, for this finite element, close to the double of the length of
 363 the local conical failure of concrete occurring just around the crack. In fact,
 364 along this zone, the bond is damaged and the stress in the reinforcement
 365 quasi constant, what corresponds to the plateau observed in figure 6. This
 366 possibility is exploited in the current paper.

367 *Reference solution compared to homogenized solution without the Helmholtz*
 368 *formulation.* To show the efficiency of the proposed method, first the Helmholtz
 369 equation is switched off, so that the homogenized solution (mesh (b) in figure
 370 5) considers a perfect bond between the reinforcement and the matrix, while
 371 the reference solution considers the possibility of sliding (mesh a). The ref-
 372 erence solution is given in Figure 6. The cracking force is reached for point
 373 A in Figure 6 (a). The crack crosses the matrix section at point B. From
 374 B to C, the crack opens and sliding occurs. For each increment of imposed
 375 displacement, the stress profile along the reinforcement is plotted in Figure 6
 376 (b). The stress concentration at mid length begins just after the crack prop-
 377 agation (between curves A and B), then the stress concentration increases
 378 until the end of loading (curve C). The solution obtained using a perfect

Table 1: Reinforced concrete tie beam material parameters

Parameter	Symbol	Value	Unit
Concrete parameters			
Young's modulus	E^m	30 000	MPa
Poisson's ratio	ν^m	0.2	—
Tensile strength in the weak zone	R_t^m	4	MPa
Compressive strength	R_c^m	57	MPa
Fracture energy	G_f^m	100	J/m^2
Reinforcement parameters			
Young's modulus	E^r	210 000	MPa
Elastic limit	f_y^r	500	MPa
Hardening modulus	H^r	1000	MPa
Interface parameters for mesh (a) in Figure 5			
Thickness	-	3	mm
Young's modulus	E^i	288	MPa
Poisson's ratio	ν^i	0.2	—
Interface parameters for mesh (b) in Figure 5			
Stiffness	H^i	40000	MPa/mm

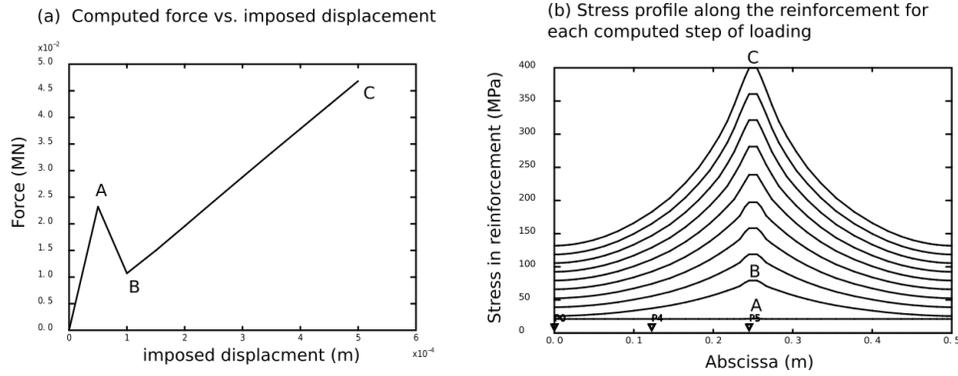


Figure 6: Reference solution (obtained with mesh(a) in Figure 5) for the tie beam with one crack: (a) force - displacement curve, (b) stress profile in reinforcement along the tie beam.

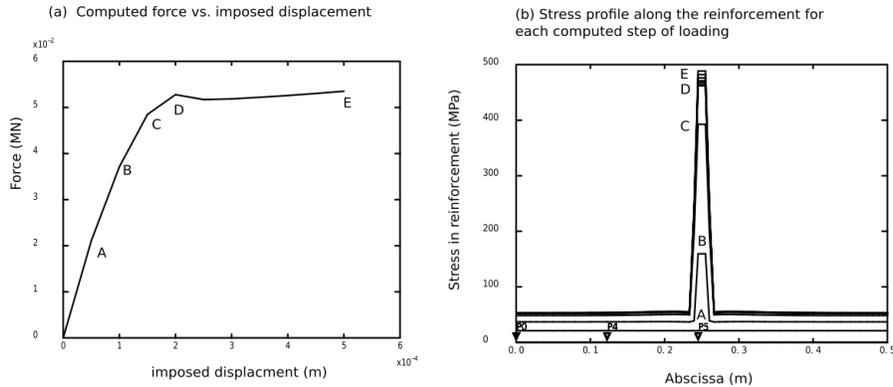


Figure 7: Solution with a perfect bond (obtained with mesh(b) in Figure 5) for the tie beam with one crack, (a) force - displacement curve, (b) stress profile in the reinforcement with the stress concentration in front of the crack.

379 bond between reinforcement and matrix is provided in Figure 7. As can be
 380 observed in 7 (a) the force-displacement curve presents greater stiffness than
 381 in Figure 6 (a). This can be explained by the greater increase of stress in
 382 the reinforcement in front of the crack. As, in this case, the reinforcement
 383 cannot slide, the reinforcement strain is concentrated in the finite element
 384 damaged by the localized crack. Consequently the strain increases faster and
 385 the plastic limit of reinforcement is reached at point D, while in the case with
 386 a possibility of sliding the plasticity of the reinforcement does not occur. This
 387 simple example shows how large the error on the assessment of a composite
 388 matrix can be when the sliding of reinforcement is neglected.

389 *Reference solution compared to homogenized solution with Helmholtz formu-*
 390 *lation.* A comparison of Figures 6 and 7 highlights the need to consider the
 391 interface behavior in a finite element analysis of composite material. The
 392 method used to consider the sliding allowed by the interface behavior led
 393 to a Helmholtz formulation that avoided meshing the reinforcement and the
 394 interface. This method, tested in the context of the homogenized formula-
 395 tion (14) uses the simplified mesh ((b) in Figure 5). The solution obtained
 396 is presented in Figure 8, where it is confronted with the reference solution.
 397 It is worth noting that the simplified solution is in good accordance with the
 398 reference one, showing that, in this case, the method based on the Helmholtz
 399 formulation is an interesting alternative to complete meshing, because it pro-
 400 vides a very close solution for a reduced meshing and a reduced computational

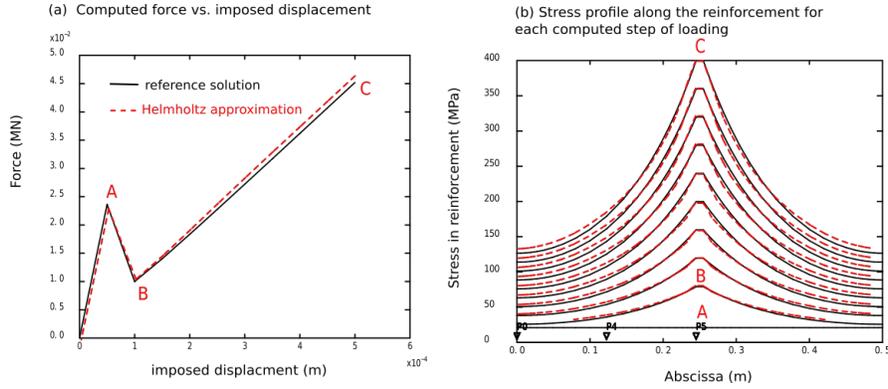


Figure 8: Comparison of the solution obtained with the simplified Helmholtz formulation (mesh (b) in Figure 5, dotted lines) with the reference solution (obtained with mesh(a) in Figure 5, plain lines) for the tie beam with one crack.

401 cost.

402 4.1.2. Theoretical case with several cracks in a reinforced concrete tie beam

403 To test the ability of the model to capture the complex phenomenon
 404 of multi-cracking, a second virtual study was carried out. It concerned a
 405 reinforced tie beam with the same cross section as in the previous case, but
 406 longer (1.5 m instead of 0.5 m), and with three prepositioned weak zones as
 407 represented and numbered in the diagram of Figure 9. All the other matrix
 408 reinforcement and interface characteristics were the same as specified in Table
 409 1. The weakest zone is in the middle of the tie, with a strength $R_t^m = 4MPa$,
 410 the second weak zone is in the left part of the tie with a strength of $1.25R_t^m$,
 411 the third in the right part with a strength of $1.5R_t^m$. The rest of the tie has
 412 a strength of $3R_t^m$. The responses of the models are compared in Figure 10
 413 in terms of force displacement and stress profiles along the reinforcement at
 414 different steps of loading. The Helmholtz results are also in good accordance
 415 with the reference solution. The cracks appear quasi simultaneously with
 416 those in the reference solution (for the same imposed displacements), and the
 417 stress levels in the reinforcement are relatively close. The small differences
 418 could be the consequences of differences of geometry: in the reference case the
 419 reinforcement was concentrated in the middle of the tie as shown in Figure
 420 9(a) while the reinforcements are assumed to be distributed homogeneously
 421 in the cross section for the homogenization method (9(b)).

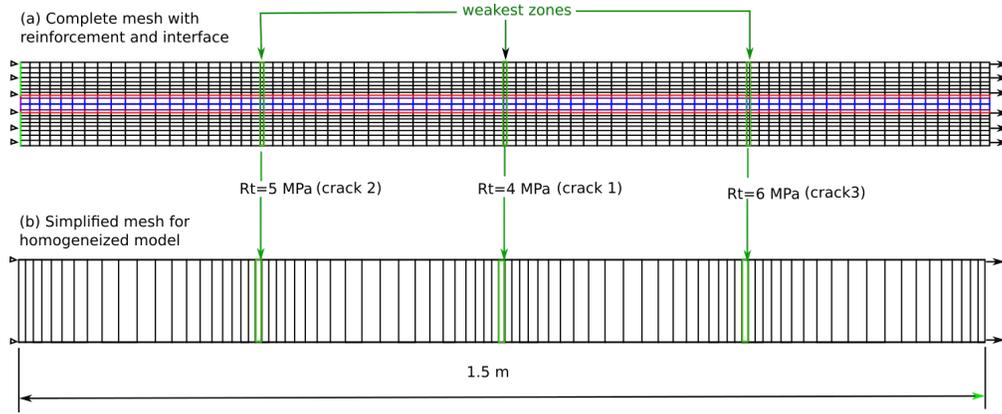


Figure 9: Meshes for the reference solution (a) with three cracks, and for the Helmholtz formulation (b)

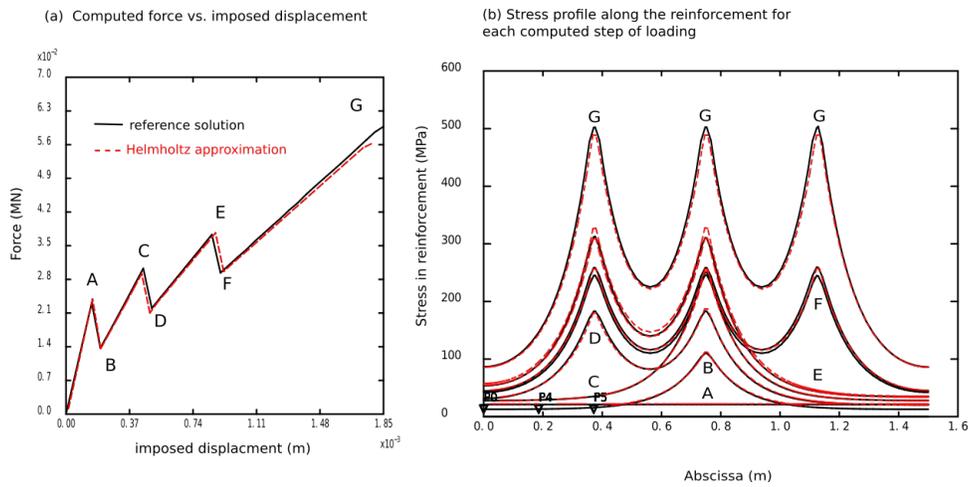


Figure 10: Comparison between the reference solution with three cracks, and the Helmholtz solution: (a) force displacement curves, (b) stresses in the reinforcement along the beam at different steps of loading.

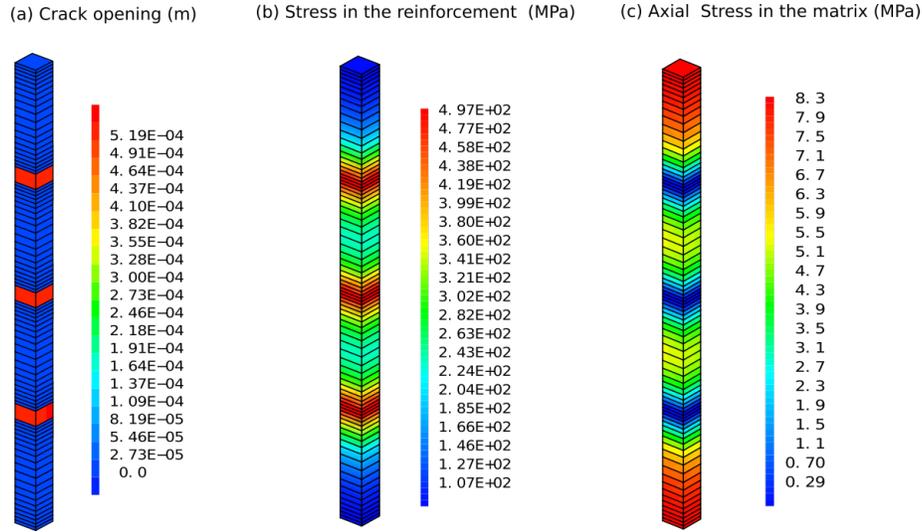


Figure 11: Illustration of results obtained with the homogenized model based on the Helmholtz formulation: (a) crack opening, (b) stress in the reinforcement, (c) stress in the matrix

422 To illustrate the fact that, in the case of the Helmholtz solution (homog-
 423 enized material case), the stresses in the reinforcement are treated as con-
 424 tinuous fields, like the stress in the matrix, Figure 11 shows the three major
 425 variables expected by the users for this type of modelling: the crack opening
 426 field, the stress in the reinforcement and the axial stress in the matrix.

427 4.2. Application to a real structure

428 The example chosen to illustrate the applicability of the method to a real
 429 structure is a reinforced concrete beam already widely studied by several
 430 authors in the framework of the French research project CEOS [8, 20]. The
 431 geometry of the beam is given in Figure 13 and the material parameters in
 432 table 2. The mesh used is very simple, it is a regular distribution of cu-
 433 bic elements shown in figure 13, independent of the steel bars position and
 434 geometry. The reinforcements are not meshed but, in accordance with the
 435 proposed method, are considered only through their areas ratios and direc-
 436 tions. In figure 13 the fields of reinforcement's area ratios are illustrated: the
 437 longitudinal re-bars are distributed in two zones, at the bottom and the top
 438 of the beam. The steel stirrups are considered also by this method with two
 439 other fields: one for the vertical parts of the frame and another one for the

440 horizontal part. The size of these zones can be chosen relatively freely, the
441 only condition to respect is that the total amount of reinforcement per zone
442 must correspond to the real one. In this example the ratios and the direc-
443 tions are defined thank to parametric fields. The parameters of each field
444 control the evolution of the fields versus the global coordinate system. So, to
445 change the reinforcement's system, only the parameters of the fields would
446 have to be changed. This method could be exploited to optimize positions,
447 directions and ratios of reinforcements without changing the mesh. In figure
448 14, the computed force displacement curve is compared to the experimental
449 one and to the results obtained with other classical models for which the
450 reinforcements are explicitly meshed [20]. This comparison allows to verify
451 that, despite the fact that reinforcements are not explicitly modelled, the
452 stiffness loss due to the progressive cracking of the concrete, and the plateau
453 of the curve predicted by the model, are close to the experimental ones and
454 perfectly compatible with the other modellings of this beam. Figure 14 gives
455 also the evolution of crack opening predicted by the model. It is worth not-
456 ing that the beam presents a multi-cracking with numerous localized crack.
457 Between 20 mm deflection and 30 mm deflection, the cracks number does not
458 evolve but their openings increase due to the sliding of longitudinal and ver-
459 tical reinforcements along the concrete. The longitudinal stress in concrete is
460 illustrated in figure 15: In figure 15, the field SNC2 is the concrete stress in
461 the axial direction. This stress reaches the compressive stress just under the
462 point of applied load, leading to a crushing of concrete in this zone, which
463 provokes a tensile stress in the upper part of steel stirrups (stress SNR1 in
464 Figure 15, in the transverse direction). Despite the localized crack observ-
465 able at mid span of the beam at the end of loading in Figure 14, the axial
466 stress in the bottom reinforcements (SNR2 in Figure 15) is not localized, this
467 is a consequence of sliding of these re-bars along the concrete in the vicinity
468 of the localized cracks. Concerning stirrups, as their diameter is four time
469 smaller than the longitudinal re-bars ones, their diffusion length (l_c in equa-
470 tion (9)) is then twofold smaller, and the stress field is less spreaded). It is
471 also worth noting that the stress representation in figure 15 allows to control
472 easily the stress levels in the different materials and the different directions,
473 simply shifting from one internal variable of the model to another one.

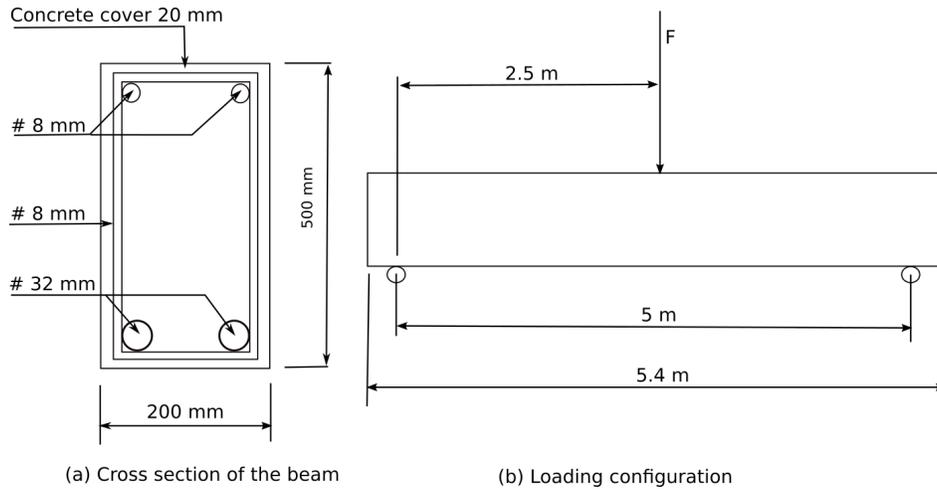


Figure 12: Reinforced concrete beam geometry

Table 2: Reinforced concrete beam material parameters

Parameter	Symbol	Value	Unit
Concrete parameters			
Young's modulus	E^m	37 200	MPa
Poisson's ratio	ν^m	0.27	–
Tensile strength in the weak zone	R_t^m	3.45	MPa
Compressive strength	R_c^m	36	MPa
Fracture energy	G_f^m	100	J/m^2
Reinforcement parameters			
Young's modulus	E^r	195 000	MPa
Elastic limit	f_y^r	466	MPa
Hardening modulus	H^r	3245	MPa

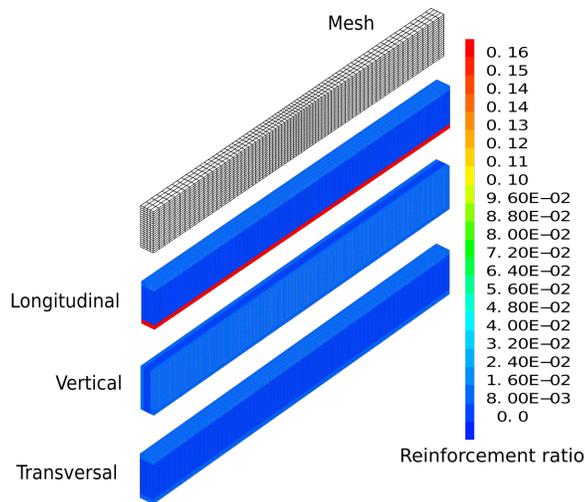


Figure 13: Reinforced concrete beam: Mesh and reinforcement ratio fields

474 5. Conclusion

475 To gain in efficiency and accuracy when calculating complex elements
 476 consisting of a mixed material combining a brittle matrix and oriented ductile
 477 fibers or reinforcements that may slip into the matrix during cracking,
 478 a non-local model of homogenized reinforcements has been developed. This
 479 model leads to the solving of a Helmholtz equation for each type of reinforce-
 480 ments. Once implemented in a finite element code, the Helmholtz equations
 481 avoid the need to mesh the reinforcements but enable their possible slid-
 482 ing to be considered, which plays an important role in the behavior of the
 483 cracked element. The paper gives the main equations and principles for the
 484 finite element implementation. It also provides the numerical solution of
 485 two theoretical tests, in order to verify that the implementation is correct.
 486 An application to a real reinforced concrete beam shows that this modelling
 487 method gives realistic responses, close to other classical models for which the
 488 reinforcement are explicitly meshed. Of course, other confrontations with
 489 experimental results will have to be done before applying the method to a
 490 real project. Perspective for continuing this work will be to improve the res-
 491 olution algorithm and to extend the model to large crack openings, to cyclic
 492 conditions (for dynamic applications [34, 29]), to short fibers that may be
 493 totally pulled out during the crack opening [16], to evolutive matrices such

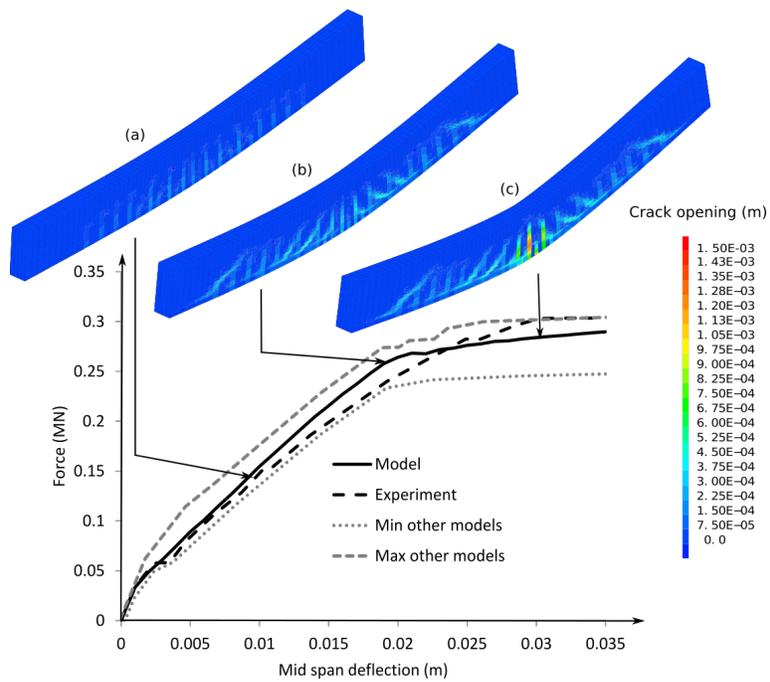


Figure 14: Force displacement curves, comparison model versus experiment, and computed crack opening [m] at different stages of loading: (a) 10 mm deflection, (b) 20 mm deflection, (c) 30 mm deflection. Min and Max of the other models come from the benchmark performed in the framework of the CEOS.fr research project [20].

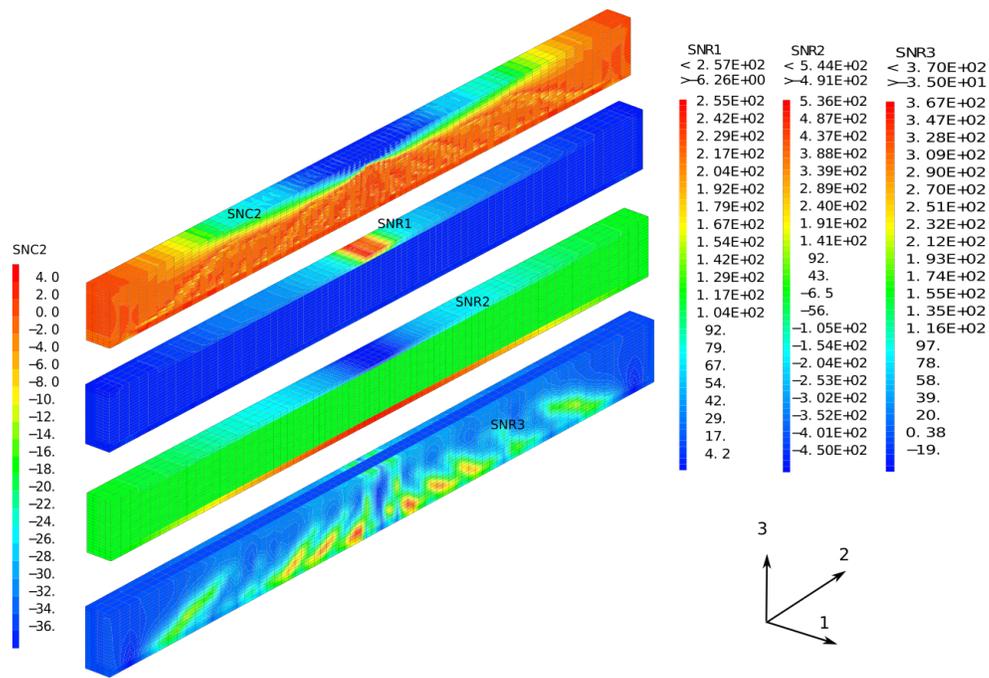


Figure 15: Computed stresses [MPa] at 30 mm deflection : SNC2 axial stress in concrete, SNR2 stress in distributed longitudinal reinforcements, SNR3 stress in distributed vertical reinforcements, SNR1 stress in distributed transversal reinforcements

494 as concrete affected by an alkali reaction [27] or delayed ettringite formation
495 [42], and to problems of reinforcement corrosion [33]. Another perspective
496 will be to consider the possible creep [40] of the interface when the loading
497 is maintained for a long period.

498 **Acknowledgements**

499 The C.E.A (French Atomic and Alternative Energies Agency) is thanked
500 for providing the finite element code Castem in its academic version. The
501 ANR (French National Research Agency) is thanked for its contribution to
502 the national research projects CEOS.fr and MACENA.

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