Determining both radial pressure distribution and torsional stiffness of involute spline couplings

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Abstract:

In this paper an analytical method is used to investigate the distortions of involute spline teeth. The following hypotheses are adopted: that teeth geometry is in conformity with standardisation, dimensions are nominal (no defect), there is no friction and the load is a pure torsional torque. Teeth distortions due to bending, shear, compression and foundation rotation are analysed. As the load is distributed along the tooth height, the displacement calculation differs from the conventional approach used for gear teeth. Sliding over the contact surfaces is also considered as it emerged during the study that this phenomenon, that has not hitherto been taken into account, plays a significant role. A punch model is used to describe the radial distribution of the contact pressure. Ascribing an arbitrary value to the tilted angle between the two contacting flanks enables the pressure profile to be evaluated, from which calculation of teeth distortions can be arrived at so as finally to obtain a new estimation of the tilted angle. Thus displacements and the contact load can be determined together by iterating the calculation procedure until convergence. Torsional stiffness, which is one of the main parameters required to predict the torque distribution along the spline coupling, is evaluated from the various displacement components. The results derived from the proposed analytical method are compared with finite element results and show good correlation.

Keywords: spline coupling, teeth stiffness, pressure distribution.

Notations:

α	Tilted angle in radians
υ	Poisson coefficient
μ	Modulus of rigidity
χ	Kolosov's constant
θ	O'Donnell's foundation deflection in radians
$\theta_{d,\, j}$	Contact angle including tooth angle at node j in radians
$\theta_f^e,\ \theta_f^i$	Shaft and sleeve tooth foundation rotation in radians
ϕ_{global}	Global rotation of the teeth in radians
ϕ_j	Contact pressure angle at node j in radians
ϕ_j	Contact angle at node j in radians
$\phi_{p,\, j}$	Contact load angularity with the force P in radians
$\phi_{p,\, j}$	Contact load angularity with the pressure $p(s_j)$ in radians
$\phi_t^e,\ \phi_t^i$	Shaft and sleeve teeth angle at the neutral axis in radians
a	Half contact length on the curvilinear axis in m
A_k	Constant factor
c	Half contact length on the x axis in m
c_φ	Torsional stiffness per unit width (along z axis) in N/rad
d	Curvilinear length in m
E	Young's modulus in Pa
F	Force applied on the contact surface per unit width in N/m
G	Shear modulus in Pa
h'	Effective height in m
$h_j^e,\ h_j^i$	Shaft and sleeve tooth height at node j in m

$h^e_{inter,j}, h^i_{inter,j}$	Shaft and sleeve tooth height at the middle of the segment j in m
h_p	Shaft and sleeve tooth height at the pitch radius in m
I_j^e, I_j^i	Shaft and sleeve tooth second moment of area at node j in m^4
k_c^e, k_c^i	Shaft and sleeve corrective coefficients related to compression
k_s^e, k_s^i	Shaft and sleeve section modulus coefficients for shear
k_0	O'Donnell's influence coefficient related to foundation rotation in N^{-1}
k_0^e, k_0^i	Shaft and sleeve influence coefficient related to foundation rotation in N^{-1}
L	Contact length on the axis x in m
l_j^e, l_j^i	Shaft and sleeve length on the axis x at node j in m
$l^e_{inter,j}, l^i_{inter,j}$	Shaft and sleeve length on the axis x at the middle of the segment j in m
ls_j	Curvilinear distance on the contact at node j in m
l_p^e	Distance between R and R_{re} in m
l_p^i	Distance between R and R_{ri} in m
M_f^e, M_f^i	Shaft and sleeve tooth foundation moment at node j per unit width in N
N	Number of teeth
n	Even number of segments
P	Circular pitch in m
$p(s_j)$	Pressure distribution along the contact line, at node j in N/m^2
$p(x)$	Pressure distribution along the axis x in N/m^2
p_r^e, p_r^i	Shaft and sleeve radial pressure in N/m^2
R	Pitch radius in m
R_b	Base radius in m
R_{ext}	External radius of the sleeve in m
R_0, R_i	Shaft and sleeve major radius in m

R_{int}	Radius corresponding to the hole of the shaft in m
R_j	Radius at node j in m
R_{re}, R_{ri}	Foundation radius of the shaft and the sleeve in m
s_i	Curvilinear coordinate at node j in m
S_j^e, S_j^i	Shaft and sleeve tooth section at node j in m^2
T	Moment applied on the contact surface per unit width in N
t_b	Circular tooth thickness at the base radius in m
T_{ext}	External torque per unit width in N
$u_{b,j}^e, u_{b,j}^i$	Shaft and sleeve tooth bending deflection at node j in m
u_c^e, u_c^i	Shaft and sleeve tooth compression deflection in m
$u_{s,j}^e, u_{s,j}^i$	Shaft and sleeve tooth shear deflection at node j in m
v_{slid}^e, v_{slid}^i	Shaft and sleeve radial displacement due to sliding in m
u_{slid}	Orthoradial displacement due to sliding in m

1. Introduction

Involute spline couplings are commonly encountered in torque transmission lines. When they are used in high technology applications, especially in aero-engine equipment, the designers have to study their behaviour under load thoroughly in order to size them accurately. Various works [1-5] have shown that the axial load distribution on the teeth is non-uniform and therefore contradicts standardisation assumptions relating to spline coupling sizing. In order to estimate the risks of fretting, wear and bearing, good knowledge of the pressure distribution on the teeth flanks is needed. This pressure distribution is mainly governed by the torsional stiffness of the joint in the axial direction and by the teeth distortions in the radial direction.

Axial torque distribution received attention from Tatur [1]. His approach was developed for straight flank couplings as shown in Fig. 1a but can also be used for any kind of spline coupling. Tatur proposed to calculate the running torque $m(z)$ transmitted from the sleeve to the shaft along the axial direction z from the following differential equation:

$$m(z)=\frac{dT^{e}(z)}{dz}=c_{\varphi}\left[\varphi^{i}(z)-\varphi^{e}(z)\right]$$

where

- $T^e(z)$ is the shaft torque (for every z, the sum of $T^e(z)$ and $T^i(z)$, the sleeve torque, being equal to the external torque T_{ext}),
- c_φ is the torsional stiffness of the joint, considered as constant whatever the z value,
- $\varphi^i(z)$ and $\varphi^e(z)$ are the twisting angles for the internal spline and external spline.

$m(z)$ is directly linked to the mean pressure acting at section z. It has been shown that this model leads to satisfactory results provided that the value assigned to the torsional stiffness c_φ is appropriate [2, 3].

A number of studies have been devoted to the evaluation of spline stiffness. Roger Ku [6, 7] developed an experimental method relating to a high-speed rotating machine spline coupling. The authors introduced dynamic coefficients, but did not propose a method to calculate the stiffness of the joint from their geometrical and material characteristics. An analytical method was developed by Marmol [8] to test rotor vibrations. He considered the shaft and sleeve teeth to behave like cantilever beams. The effects of bending, shear and compression were considered, and the rotation at the tooth foundation centre was also taken into account. But the proposed solution departs from reality since the contacts between sleeve and shaft were considered to be merely punctual. Another simplified analytical method was developed by Hayashi [9] where teeth of external and internal splines were considered to have a rectangular shape and the only teeth deflection studied was bending.

Tooth behaviour will vary according to whether the load is introduced as punctual or distributed. Models need to consider a radial pressure distribution to obtain more accurate results. In 1969, Tatur [1] posited a uniform radial pressure distribution while, in 1982, Volfson [4] assumed a parabolic distribution, but neither was able to validate their hypotheses. With modern hardware and software advances in finite element calculations, we can now predict pressure distribution in an appropriate manner. Adey [5] and Leen [10] have developed FE analyses with experimental validations illustrating that radial pressure distribution is far from being uniform and is actually close to that of the punch model. But FE approaches are relatively costly, with each new study requiring the development of an accurate FE model.

The difficulty encountered by designers seeking to use Tatur's model lies in the fact that the literature does not offer any general method to calculate at low cost a suitable value for spline stiffness c_φ and, in conjunction, a good prediction of radial pressure distribution that caters for the necessary relationships linking these two parameters.

The present paper aims to create an analytical method dedicated to determining torsional stiffness of standard involute spline couplings as shown in Fig. 1b. This method seeks to consider the interdependence between teeth distortions and pressure distribution in its calculations. Firstly, the punch model selected to represent radial pressure distribution will be described. The general process leading to the calculation of torsional stiffness will then be introduced. Developments relating to the calculation of pressure distribution and the various teeth deflections will then be set forth in detail. Finally, to validate the analytical approach, the results thus obtained will be compared with FE results.

2. Choosing a model for teeth contact

To understand the contact phenomenon in a spline coupling, Leen [10] developed an experiment similar to a punch test. The experimental measurements were compared on the one hand with FE results and on the other with the analytical model associated with the punch shown in Fig. 2a (Hanson [11, 12]). In spline couplings, the sleeve tooth can be considered to be the punch and the shaft tooth the plane, or vice versa. However, Hanson's punch model considers a symmetric pressure distribution, whereas pressure distribution on a spline tooth is dissymmetric [10].

Various other works related to punch models were considered in the quest for a model capable of introducing a dissymmetric distribution. Sackfield [13] used a half plane formulation to analyse the pressure and slip of a tilted punch as shown in Fig. 2b-c. Unlike with Hanson, the punch load is a force and a torque with two configurations. In the first configuration, the load is centred on the punch and the contact is complete (as in Fig. 2b). In the second configuration, the load is off-centre, imposing a receding contact (as in Fig. 2c). The equation for pressure distribution *p(x)* along the contact surface in the complete contact configuration is

$$p(x) = \frac{1}{\sqrt{(c^2 - x^2)}}\left(\frac{F}{\pi} + \frac{\alpha}{\mathrm{A_K}}x\right),$$

where

- F is the external force applied on the punch per unit width (along the z axis),
- $\mathrm{A_K} = \frac{\chi + 1}{4\,\mu}$, with χ is Kolosov's constant and μ the modulus of rigidity,
- α is the tilted angle,

- c represents the half contact length,
- x is a variable parameter, $x \in [-c\ ;\ c]$.

The applied torque T is obtained by the equation

$$T = \frac{c^2\, \pi\, \alpha}{2\, A_K} \tag{1}$$

More recently Goryacheva [14] developed an analytical method for the inclined punch having a flat base and blend radii as shown in Fig. 2d.

Finally, taking into account Leen's study and the geometry of spline couplings, which does not present blend radii at the teeth ends, Sackfield's model can be considered to be the most appropriate. The tilted angle α is the one formed by the two surfaces that are punching each other as shown in Fig. 3. α is linked to the global rotation of the sleeve related to the shaft, which is considered to be fixed. This rotation is the result of the shaft and sleeve teeth distortions. Consequently, the punch model cannot be applied immediately to the spline coupling and the various sources of teeth distortion need to be studied.

3. Defining the main calculation process

Fig. 4 describes the general process that will allow teeth distortions, radial pressure distribution and spline torsional stiffness to be obtained. This process starts with a loop structure including three different steps. The aim of this first part is to determine the value of the tilted angle α and, by so doing, distortions and pressure. Indeed, arbitrarily giving an initial value equal to zero to the tilted angle α makes it possible to develop a first calculation step where the pressure profile arising from this angle α can be evaluated using the Sackfield model. Knowing the distributed load acting on the contact surfaces, a second step is dedicated to teeth distortion calculation. Subsequently, a third step can then generate a new estimation of the tilted angle from the displacements related to distortions. Thus, the required value of the tilted angle is determined by iterating this calculation procedure until convergence. The second part of the process is limited to evaluating torsional stiffness through knowledge of the various displacement components. All the steps in this process are described below in a more detailed manner.

4. Calculating pressure distribution

This section explains how to determine the radial pressure distribution considering the external torque T_{ext} and the tilted angle α to be known. Only study of the external spline is presented. Radial pressure at point P_1 can be represented (P_1 is the crossing point between the contact curve and the pitch diameter) by a torque T and a force F. Fig. 5 shows loading of the shaft.

T can be obtained from equation (1). The equilibrium law applied to the external spline on the shaft section centre gives the resulting equivalent force F :

$$F = \left(\frac{T_{ext}}{N} + T\right)\frac{1}{R_b} \qquad (2)$$

where

- T_{ext} is the torque per unit width applied on the shaft,
- N is the number of teeth,
- T is the torque per unit width applied by the sleeve to the shaft at the tooth contact surface,
- R_b is the base radius of the spline.

The Sackfield pressure distribution equation cannot be applied directly since the punch surface is considered to be planar while the tooth surface is curved. The linear variable x is replaced by a curvilinear variable *s*. Furthermore, the tooth section is variable, hence the tooth is divided along its height into *n* segments as shown in Fig. 6 . Thus *p(x)*, which becomes $p(s_j)$, is calculated for each s_j, where *j* varies from 0 to *n*.

The pressure distribution expression becomes:

$$p(s_j) = \frac{1}{\sqrt{(a^2 - s_j{}^2)}} \left(\frac{F}{\pi} + \frac{\alpha}{\mathrm{A_K}} s_j \right) \tag{3}$$

where *a* is the half length of the contact along the curvilinear axis and s_j is the curvilinear coordinate $s_j = \sum_{k=1}^{j-1} ls_k - \alpha$. ls_k is the curvilinear distance between the contact start radius R_i, and node *k*. During this calculation, the approximation that the tooth spline curve can be cut into a straight segment between two consecutive nodes is adopted. Hence the curvilinear distance ls_k is given as

$$ls_k = \sqrt{\left(\frac{h_k^e - h_{k+1}^e}{2} \right)^2 + \left(\frac{L}{n} \right)^2}$$

where

- *L* is the contact length in the *x* direction, $L = R_o - R_i$,

  where

  - $R_i = \dfrac{P(N-1)}{2\pi}$,
  - $R_0 = R_i + \dfrac{P}{\pi}$,

- $h_k{}^e$ is the tooth height of the shaft at node *k*

  where

  - for *k* = 0, at the tooth base , $h_0^e = 2\pi \dfrac{R_{re}}{N}$,

- for k = 1 to n, $h_k^e = 2\ R_k\ \sin\left(\theta_{d,k}\right)$,

where

- $\theta_{d,k}$ is the contact angle including the tooth angle (Cornell [15])

$$\theta_{d,k} = \frac{t_b}{2\ R_b} - \tan\left(\phi_k\right) - \phi_k\ ,$$

where

- ϕ_k is the contact pressure angle $\phi_k = \operatorname{asin}\left(\frac{R_b}{R_k}\right)$,
- t_b is the circular tooth thickness at the base radius,

- R_k is the radius at node k, $R_k = R_i + \left(\frac{k}{n}\right)\frac{P}{\pi}$.

This step shows that α appears to be the key parameter required to solve the problem. This angle depends directly on the behaviour of the teeth, hence the next step involves determination of distortions.

5. Calculating teeth distortions

Since the teeth geometry of the external spline is close to that for spur gears, the literature has been analysed to take note of all phenomena addressed in spur gear teeth deformation studies. Terauchi [16] and Oda [17, 18] defined tooth stiffness in relation to two dimensional elasticity theory and a “mapping function”. This function allows an equation to be obtained that divides the tooth profile after deformation. This equation is based on Timoshenko’s work, which uses 14 variables. Among the studies encountered, Cornell’s appeared to provide the basis for several other works. Cornell [15] developed an analytical method allowing stiffness of the teeth to be found and compliance and stress sensitivity to be determined. In this model, the distortions of teeth spur gears are due to bending, shear, compression and foundation rotation. Analysis of this last parameter is based on the work of O'Donnell [19, 20]. Huang [21] proposed an analytic dynamic analysis of a spur gear. Bending, shear, local compression due to the contact between spur gears, and foundation rotation are also considered to determine stiffness. Cornell's [15] and Matusz's [22] publications are referred to in order to calculate foundation rotation. More recently, Sainsot [23] used the theory of elasticity to model spur gear body distortion. In this publication, tooth stiffness is also based on Cornell's theory. Experimental methods [24] or finite element investigations [25] addressed the question of evaluating teeth distortions. The results proposed in these papers were useful in validating the assumptions made in studies devoted to spur gears but cannot be used for spline coupling. Indeed, as the nature of the contact and the geometry of the internal tooth are different, the values for deformations too are necessarily different.

Many of the numerical analyses carried out within the framework of the present study showed that the slip phenomenon that takes place at the contact surface has significant consequences. Indeed, the flank is inclined and pressure exerts radial forces that impose a compression of the shaft body and an expansion of the sleeve body. This generates orthoradial displacements, which can have a major influence. In order to obtain a good estimation of the teeth stiffness, sliding needs to be taken into account in stiffness calculation.

Finally, according to these different works, teeth behaviour can be considered to be a superposition of four phenomena:

1. bending and shear of the teeth,
2. compression,
3. tooth foundation rotation,
4. sliding.

These various phenomena will now be described separately.

5.1. Bending and shear of the teeth

According to a number of papers, shear force and bending moment appear to be the principal effects on deflection of the teeth. These phenomena are based on stress analysis and can thus be expressed easily.

Fig. 6 provides an illustration of the following explanations. The teeth on sleeve and shaft can be divided into two sections. The first, for the external spline, is between the shaft foundation radius R_{re} and the sleeve minor radius R_i, and that for the internal spline between the sleeve foundation radius R_{ri} and the shaft major radius R_0. On these parts, no loading is applied. The second is between R_i and R_0, where contact is made. As shown in the pressure distribution formula, the geometry imposes a division of the tooth into segments.

For shear and bending deflections, the teeth are considered to behave like cantilever beams. The shear deflection of the sleeve, $u^i_{s,j}$, or of the shaft, $u^e_{s,j}$, is the sum of the deflection on each segment of the tooth.

<u>Note</u>: when a mathematical expression is identical for the shaft and the sleeve, the index ξ replaces indices "*e*" and "*i*".

The shear deflection expression at node *j* is

$$u^{\xi}_{s,j} = \frac{k^{\xi}_s}{G} \sum_{k=0}^{j} \frac{V^{\xi}_k}{S^{\xi}_k} x_k \,, \qquad (4)$$

where

- V_k^{ξ} is the shearing force per unit width, the integral of the pressure along the contact surface between the fixed extremity and node *k*

$$V_k^e = \sum_{q=0}^{k} \left(\frac{F}{\pi} \operatorname{asin}\left(\frac{s_q}{a}\right) - \frac{\alpha}{A_K} \sqrt{a^2 - s_q^2} \right)$$, and

$$V_{n-k}^i = V_k^e,$$

- S_k^ξ is the surface area, where V_k^ξ is applied. As unit width is considered, this surface area depends of the tooth height before and after the segment, h_k and h_{k+1}. According to Cornell [15], calculating S_k^ξ using the following formulae improves accuracy

$$S_k^\xi = \frac{2}{\frac{1}{h_k^\xi} + \frac{1}{h_{k+1}^\xi}} = 2 \frac{h_k^\xi \, h_{k+1}^\xi}{h_k^\xi + h_{k+1}^\xi},$$

where

  - h_k^e is the shaft tooth height, already defined in the pressure distribution description. For the sleeve the height is
    - $h_0^i = 2\pi \frac{R_{ri}^i}{N}$ for $k = 0$
    - $h_{n-k+1}^i = 2 R_j \sin\left(\frac{\pi}{z} - \theta_{d,k}\right)$ for k from 1 to n,

- x_k is the distance between the considered node k and the previous k - 1 in the x direction,

- k_s^ξ is the section modulus coefficient.

The bending deflection of the sleeve and shaft, $u_{b,j}^\xi$, at node j, is calculated using the second derivative of the bending moment

$$\left(u_{b,j}^\xi\right)'' = \sum_{k=0}^{j} \frac{M_{b,k}^\xi}{E\, I_k^\xi}, \qquad (5)$$

where

- $M_{b,k}^{\xi}$ is the bending moment per unit width, which can be expressed by

$$M_{b,k}^{\xi}=\sum_{o=0}^{k}\sum_{q=o}^{k}\left(\frac{f_q^{\xi}}{\mathrm{E}\,I_o^{\xi}}\left(\left(l_{inter,q}^{\xi}-l_o^{\xi}\right)\cos(\theta_q^{\xi})-\frac{h_{inter,q}^{\xi}}{2}\sin(\theta_q^{\xi})\right)\right),$$

where

  - f_q^{ξ} is the force per unit width applied on segment q

$$f_q^{\xi}=\left(\frac{F}{\pi}\left[\operatorname{asin}\left(\frac{s_{q+1}}{a}\right)-\operatorname{asin}\left(\frac{s_q}{a}\right)\right]-\frac{\alpha}{\mathrm{A_K}}\left[\sqrt{a^2-s_{q+1}^2}-\sqrt{a^2-s_q^2}\right]\right),$$

  - I_o^{ξ} is the second moment of area expressed where the moment is applied. According to Cornell [15] and working per unit width, I_o^{ξ} can be expressed

$$I_o^{\xi}=\frac{2}{\dfrac{1}{h_o^{\xi^3}}+\dfrac{1}{h_{o+1}^{\xi^{\ 3}}}}=\frac{\left(h_o^{\xi}\ h_{o+1}^{\xi}\right)^3}{6\left(h_o^{\xi^3}+h_{o+1}^{\xi^{\ 3}}\right)},$$

  - l_o^{ξ} is the distance between the foundation, the fixed tooth extremity and the point where the moment is applied in the x direction

$$l_o^{\xi}=\frac{o-1}{n-1}\left(R_0-R_i\right)+l_a^{\xi},$$

  where

    - $l_a^e=R_i-R_{re}$,
    - $l_a^i=R_{ri}-R_0$,

  - $l_{inter,q}^{\xi}$ is the distance between the foundation, the fixed tooth extremity, and the point where the force f_q^{ξ} is applied in the x direction

$$l_{inter,q}^{\xi}=\frac{l_q^{\xi}+l_{q+1}^{\xi}}{2},$$

- $h_{inter,q}^{\xi}$ is the height of the tooth, where the force f_q^{ξ} is applied in the y direction

  $h_{inter,q}^{\xi} = \dfrac{h_q^{\xi} + h_{q+1}^{\xi}}{2}$.

To conclude, the deflection due to bending and shear is the sum of $u_{b,j}^{\xi}$ and $u_{s,j}^{\xi}$. To find the section modulus coefficients of the shear deflection, a confrontation between analytical results and finite elements was performed. Using FE analysis, it is impossible to separate the effects of bending and shear so the comparison must take both into account. The 2D FE model used to determine these coefficients is given in section 8.1 (Fig. 10). Introducing the torque at the sleeve external diameter and clamping the shaft root diameter, it is possible to obtain the coefficient k_s^e, the tooth foundation rotation being annihilated in such a configuration. Deflection is evaluated along the symmetry axis of the shaft tooth. An equivalent approach where the sleeve root diameter is clamped and the shaft internal diameter is loaded allows k_s^i to be calculated. These evaluations were made for a large set of spline couplings (different pitches and different tooth numbers), and for various torques, leading always to the same value of the section modulus coefficients: $k_s^e = 0.94$ and $k_s^i = 0.8$. Fig. 7 shows the results obtained with: P = 7.85mm, N = 18 teeth, R_{ext} = 35mm, R_{int} = 2mm and T = 7.2N, E = 207GPa and G = 80.1GPa.

5.2. Teeth compression

The compression equation derives from Marmol's study. u_c^{ξ} is the compression deflection on the contact at pitch diameter

$$u_c^{\xi} = k_c^{\xi} \frac{f_j^{\xi}\ h_j^{\xi}}{d\ \mathrm{E} \cos(\theta_j^{\xi})}, \tag{6}$$

where

- $j = n/2$,

- k_c^{ξ} is a corrective coefficient,

- d is the curvilinear length, where f_j^{ξ} is applied $d = \frac{s_{j+1} - s_{j-1}}{2}$.

To find the corrective coefficients of the compression deflection, a comparison between analytical results and finite elements was performed leading to $k_c^e = 0.531$ and $k_c^i = 0.406$.

5.3. Foundation teeth rotation

Distortion creates a foundation rotation. This phenomenon has been studied by a number of researchers, assuming different stress distributions. One of these studies was developed by O'Donnell [16, 17], who found a realistic solution based on a cubic distribution of stress.

According to O'Donnell, the foundation deflection θ is valid only on the beam support, near the neutral axis.

$$\theta = \frac{16.67\,(1-\nu^2)\,M_f}{\pi\,\mathrm{E}\,h'^2} + \frac{(1-\nu-\nu^2)V}{\mathrm{E}\,h'}$$

where

- h' is the effective height, which is equal to 1.5 times the beam height,
- θ is the deflection of a cantilever due to the elasticity support,
- V is the shear load at the support per unit width,
- M_f is the torque at the support per unit width.

It is then possible to apply this theory to the foundation rotation of the spline. The foundation rotation becomes:

$$\theta_f^{\xi} = \frac{k_0^{\xi}\,M_f^{\xi}}{h_j^{\xi\,2}} + \frac{(1-\nu-\nu^2)V_j^{\xi}}{\mathrm{E}\,h_j^{\xi}} \text{ with } j = 0, \qquad (7)$$

where

- M_f^{ξ} is the foundation moment per unit width for the shaft or the sleeve. This moment M_f^{ξ} is found by applying the equilibrium law to the shaft or the sleeve at the crossing point P_2, between the mid-tooth height and the foundation diameter as shown in Fig. 5,

  $$M_f^{\xi} = F\left(-\frac{h_p}{2}\sin\phi_{p,j} + l_p^{\xi}\cos\phi_{p,j}\right) - T \quad \text{with } j = n/2,$$

  where

  - h_p is the tooth height at the pitch radius. This is the same both for the sleeve and the shaft

    $$h_p = \frac{P}{\pi} N \sin(\theta_{d,j}), \text{ with } j = n/2,$$

  - l_p^{ξ} is the distance between the foundation radius and the pitch radius

    $$l_p^e = R - R_{re} \text{ and}$$

    $$l_p^i = R_{ri} - R,$$

  - $\phi_{p,j}$ is the contact load angularity with pressure $p(s_j)$, $\phi_{p,j} = \pi - \theta_{d,j}$,

- k_0^{ξ} is the influence coefficient. The value given by O'Donnell, for a half plane support is

  $$k_0 = \frac{16.67(1-\nu^2)}{\pi E h'^2},$$

  This value is not available for the spline coupling. Comparisons between FE model results and analytical model results allow this influence coefficient for the shaft and the sleeve to be determined

  $$k_0^e = \frac{6(1-\nu^2)}{\pi E h'^2} \text{ and}$$

$$k_0^i = \frac{8(1-\nu^2)}{\pi\, \mathrm{E}\, h'^2}\,.$$

5.4. Sliding

Sliding has never before been taken into account in determining spline coupling torsional stiffness. Sliding can be calculated using a comparison between this phenomenon and the radial compression / extension of a hollow shaft. Indeed, the radial component of the contact teeth pressure creates a body compression of the shaft and a body expansion of the sleeve as shown in Fig. 8. These new radial displacements lead to an orthoradial displacement along the contact surface of the teeth.

To solve this problem, the shaft and the sleeve are considered to be a hollow shaft. For the external spline, the maximum radius is the foundation radius of the shaft teeth, R_{re}. For the internal spline, the minimum radius is the foundation radius of the sleeve teeth, R_{ri}. The radial component of the contact pressure is distributed uniformly over the outside diameter for the shaft and inside diameter for the sleeve. For the shaft, the resulting equivalent pressure p_r^e is

$$p_r^e = \frac{F\,\sin(\phi_{p,j})}{2\,\sin\left(\frac{\pi}{\mathrm{N}}\right) R_{re}} \quad \text{with } j = n/2,$$

for the sleeve the resulting equivalent pressure p_r^i is

$$p_r^i = \frac{F\,\sin(\phi_{p,j})}{2\,\sin\left(\frac{\pi}{\mathrm{N}}\right) R_{ri}} \quad \text{with } j = n/2.$$

The radial displacement created by the sliding phenomenon is a superposition of shaft and sleeve displacements. From Lame’s equations, the shaft and sleeve radial displacements are

$$v_{slid}^{e} = \frac{p_r^e\ R_{re}}{\mathrm{E}\left(R_{re}{}^2 - R_{int}{}^2\right)}\left(\left(1-\upsilon\right)R_{re}{}^2 + \left(1+\upsilon\right)R_{int}{}^2\right),$$

$$v_{slid}^{i} = \frac{p_r^i\ R_{ri}}{\mathrm{E}\left(R_{ext}{}^2 - R_{ri}{}^2\right)}\left(\left(1-\upsilon\right)R_{ri}{}^2 + \left(1+\upsilon\right)R_{ext}{}^2\right),$$

where

- R_{ext} is the external radius of the sleeve,
- R_{int} is the hole radius of the shaft.

Finally these displacements create an orthoradial displacement at the pitch diameter

$$u_{slid} = \left(v_{slid}^{e} + v_{slid}^{i}\right) \tan\left(\phi_{\mathrm{j}} + \frac{\frac{\pi}{N} + 2\,(\tan(\phi_j) - \phi_j)}{4}\right) \text{ with } j = n/2. \qquad \textbf{(8)}$$

6. Defining tilted angle

The tilted angle is the angle between the contact surface of the sleeve and the shaft. For readier comprehension, the shaft and sleeve deformations are separated.

One method to obtain the teeth rotations is to assume that the deflection of the contact line is the same at deflection of the neutral axis. Fig. 9a shows a tooth sleeve before and after loading. After distortion, the extremity points of the contact surface C and D become C' and D' and form a line Δ'^{i}. The angle between the line (CD) and the line Δ'^{i} is θ_t^i. Only the deflections determined on the neutral axis are taken into account in the θ_t^i determination

$$\theta_t^i = \frac{u_{s,n}^i + u_{b,n}^i}{R_i - R_0} + \theta_f^i\ .$$

Fig. 9b shows the shaft before and after loading. The deformation imposes a rotation θ_t^e, which is the rotation of the line Δ^e

$$\theta_t^e = \frac{u_{s,n}^e + u_{b,n}^e}{R_i - R_0} + \theta_f^e \ .$$

These rotations and the local distortion on the contact (sliding and compression) create a body rotation of the sleeve. This global rotation θ_{global}, that does not appear on the figures, transforms the line Δ'^i to Δ^i. The expression of the global rotation is

$$\theta_{global} = \left(u_{\ slid} + u_c^i + u_c^e + u_{s,n/2}^i + u_{s,n/2}^e + u_{b,n/2}^i + u_{b,n/2}^e + \frac{\theta_f^i}{l_p^i} + \frac{\theta_f^e}{l_p^e} \right) \frac{2\pi}{P\ N} \ .$$

The tilted angle α is the angle between Δ^e and Δ^i. $\alpha = \theta_t^e - (\theta_t^i + \theta_{global})$. **(9)**

7. Determining torsional stiffness

All the parameters for torsional stiffness determination are defined. It is now possible to formulate precisely torsional stiffness c_φ as

$c_\varphi = \dfrac{T_{ext}}{\theta_{global}}$ in [N/rad]. **(10)**

8. Finite element modelling and results

A comparison between analytical results and 2D FE results will now be given. The spline dimensions are standard and the sleeve external diameter is chosen to obtain a second moment of area bigger than the shaft's. To scan different configurations, three different circular pitches, three teeth numbers and various outside torques are taken into account.

8.1. Description of finite element models

In order to validate the study, two-dimensional plane strain finite element models were made of different involute spline couplings and Abaqus was used to perform calculation. The meshing used to model the two parts constitutive of the coupling, the shaft and sleeve, are shown in Fig. 10. The final model is the sum of these two parts. The boundary conditions are defined as follows: contact is specified on the teeth flanks (but no friction), nodes located on the shaft internal diameter are locked but their radial expansion is kept possible and torque is applied to the external edge of the sleeve. Cyclic symmetry conditions are used on the two lateral edges to simulate the behaviour of the complete spline coupling.

8.2. Adequacy between analytical model and FE model:

Fig. 11 shows the comparison between finite element results and analytical model results. The characteristics of the involute spline coupling considered in this example are: P = 3.93mm, N = 18 teeth, R_{ext} = 17.5mm, R_{int} = 1mm, Text = 5N, E = 207GPa and G = 80.1GPa.

The variations between both models are shown in Tab. 1, the reference for variations being the analytical model. The two nodes located at the ends of the contact edge are not considered in this comparison because the analytical pressure is theoretically equal to infinity when $s = a$ or $s = -a$ (See Eq. 3 and Fig. 11). Using FE models, such values would be approximated only by significantly refining the meshing. The average variation, calculated on the remaining contact length, which corresponds to 90% of the entire length, is 3%. The maximal variation is about 6%.

The analytical model's precision is then sufficient to consider that radial pressure distribution is correctly calculated.

Tab. 2 allows the analytical torsional stiffness and the FE results for different spline couplings to be compared, the characteristics of the material being the same as those of the model presented above. In this table, the finite element torsional stiffness is calculated by dividing the external torque by the difference between the rotations at the foundation radius of the sleeve and the shaft. The reference used to evaluate variations is the analytical model.

Tab. 2 shows precision of the analytical model. Indeed precision on the torsional stiffness is greater than 93%.

For the same couple of circular pitch and number of teeth, teeth stiffness is identical whatever the torque applied, this being normal and proving that the analytical model gives coherent results.

The different phenomena involved in the problem do not have the same influence on stiffness or pressure distribution. The contributions made by the various phenomena on the torsional stiffness value for the spline coupling presented above in this section are:

- 58% is due to sliding,
- 29% is due to the combined effect of shear and bending,
- 8% is due to compression,
- 5% is due to foundation rotation.

From these contributions it is now possible to analyse the influence of the different corrective coefficients used in the analytical model. Sliding was evaluated without using the corrective coefficient. k_s^{ξ} shear and bending coefficients appeared to be stable during all the various tests carried out. Modelling of these three phenomena, which account for 87% of torsional stiffness, is thus extremely robust as it is not dependent on variations in corrections. The values provided above for k_c^{ξ} compression coefficients and k_o^{ξ} foundation rotation coefficients are average values. Indeed, these coefficients may vary slightly with the geometry and the materials considered in the problem. But their minor variations have only a slight effect on the total value for stiffness since the phenomena concerned account for only approximately 13% of the total.

A confrontation between the results given by the analytical model and the results given by 3D FE analyses was also conducted. These studies showed that good correlations are achieved between results, especially at the ends of the coupling that are the most critical zones.

9. Conclusion

Sackfield's punch model was combined with more conventional teeth distortion analyses to evaluate analytically the pressure distribution taking place between the teeth of involute spline couplings. The field of validity of the proposed method is defined by the following assumptions: that teeth geometry is in conformity with standardisation, dimensions are nominal (no defect), there is no friction and the load is a pure torsional torque.

Whereas previous studies have attempted to find formulae for teeth stiffness and pressure distribution separately, this paper takes both phenomena into account to manage their mutual influence. A new analytical method to define both spline torsional stiffness and radial pressure distribution is thus proposed. It considers different teeth distortions, due to bending, shear, compression, foundation rotation, and sliding at contact.

The results obtained from the study are compared with 2D FE models. These confrontations show that a good estimation of teeth stiffness can be found analytically. They also prove that the phenomena have varying degrees of influence on the stiffness value. Sliding, which has until now never been taken into account, appears to be the most significant parameter in determining torsional stiffness and pressure distribution.