



HAL
open science

Dynamics of two vibro-impact nonlinear energy sinks in parallel under periodic and transient excitations

Tao Li, Etienne Gourc, Sébastien Seguy, Alain Berlioz

► To cite this version:

Tao Li, Etienne Gourc, Sébastien Seguy, Alain Berlioz. Dynamics of two vibro-impact nonlinear energy sinks in parallel under periodic and transient excitations. *International Journal of Non-Linear Mechanics*, 2017, 90, pp.100-110. 10.1016/j.ijnonlinmec.2017.01.010 . hal-01820032

HAL Id: hal-01820032

<https://hal.insa-toulouse.fr/hal-01820032>

Submitted on 3 Dec 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 Dynamics of two vibro-impact nonlinear energy sinks in 2 parallel under periodic and transient excitations

3 T. Li^a, E. Gourc^b, S. Seguy^a, A. Berlioz^a

4 ^a*Université de Toulouse, Institut Clément Ader (ICA), CNRS-INSA-ISAE-Mines*
5 *Albi-UPS, Toulouse, France*

6 ^b*Space Structures and Systems Laboratory, Department of Aerospace and Mechanical*
7 *Engineering, University of Liege, 1 Chemin des Chevreuils (B52/3), B-4000 Liege, Belgium*

8 **Abstract**

9 A linear oscillator (LO) coupled with two vibro-impact (VI) nonlinear energy
10 sinks (NES) in parallel is studied under periodic and transient excitations, respec-
11 tively. The objective is to study response regimes and to compare their efficiency
12 of vibration control. Through the analytical study with multiple scales method,
13 two slow invariant manifolds (SIM) are obtained for two VI NES, and different
14 SIM that result from different clearances analytically supports the principle of
15 separate activation. In addition, fixed points are calculated and their positions are
16 applied to judge response regimes. Transient responses and modulated responses
17 can be further explained. By this way, all analysis is around the most efficient
18 response regime. Then, numerical results demonstrate two typical responses and
19 validate the effectiveness of analytical prediction. Finally, basic response regimes
20 are experimentally observed and analyzed, and they can well explain the com-
21 plicated variation of responses and their corresponding efficiency, not only for
22 periodic excitations with a fixed frequency or a range of frequency, but also for
23 transient excitation. Generally, vibration control is more effective when VI NES
24 is activated with two impacts per cycle, whatever the types of excitation and the

25 combinations of clearances. This observation is also well reflected by the separate
26 activation of two VI NES with two different clearances, but at different levels of
27 displacement amplitude of LO.

28 *Keywords:* Vibro-impact, Targeted energy transfer, Nonlinear energy sink,
29 Impact damper

30 **1. Introduction**

31 The excessive vibration energy of a targeted system can be dissipated by using
32 an auxiliary device, which absorbs and dissipates such undesired energy by pro-
33 ducing a force opposing it, either continuously such as in the case of nonlinear en-
34 ergy sink (NES) with continuous nonlinearity [1, 2, 3, 4, 5] or segmentally by NES
35 with piece-wise nonlinearity [6] or intermittently and instantly by impact damper
36 [7]. Due to its simplicity of construction, fast response at the initial stage of per-
37 turbation and effectiveness at a broadband frequency, impact damper has been
38 studied several decades ago [8, 9], no matter the case with only one impact pair
39 [10, 11, 12, 13, 14] or multi-units with several impact-pairs [15, 16, 17, 18, 19, 20].

40 Recently, impact damper is studied further and more clear under the context
41 of targeted energy transfer (TET) [21, 22] and termed as vibro-impact (VI) NES
42 [23, 24, 25]. Its mechanism of TET is revealed by studying its Hamiltonian system
43 [26], and it is observed that some special orbits are responsible for the irreversible
44 transfer of energy from a main system to an attached VI NES. Through directly
45 studying the system with damping, a slow invariant manifold (SIM), which is
46 firstly used for the analysis of a NES with cubic nonlinearity [27, 28], can be
47 obtained by multiple scales method and be applied to analyze response regimes
48 [29, 30, 31, 32, 33]. As a result, transient responses and unsteady responses such

49 as chaotic strongly modulated response (SMR) can be well explained for the first
50 time [31, 34]. Dynamics, such as response regimes and bifurcations, is compre-
51 hensively re-analyzed around SIM [35], more specifically, around the regime with
52 two impacts per cycle. In [36], the efficiency of different response regimes around
53 SIM is compared, and the response regime with two impact per cycle and around
54 the entrance of SMR is found to be optimal. Based on this efficiency compari-
55 son of response regimes, a common optimization design criterion of VI NES is
56 proposed for different excitation types, and the central idea is to make the most
57 efficient response regime with two impacts per cycle exist or last as long as pos-
58 sible. Therefore, it is natural to expect the following design criterion for multi VI
59 NES: to make each VI NES activated with this optimal response regime.

60 About its application, any analytical study of nonlinear systems coupled with
61 NES will be difficult, no matter this nonlinear system is a turning system [37]
62 or a helicopter system [38], or even a simple rod [39]. For VI NES, a solution
63 is to found its activation condition without considering the specific type of a tar-
64 geted system. It is found that the activation of a VI NES is limited to a range of
65 displacement amplitude of a linear system [36]. From the viewpoint of energy,
66 it means that its effectiveness is not only limited by a minimal value (activation
67 threshold) but also a maximal value. Therefore, it is natural to increase the range
68 of effectiveness of vibration control by designing different VI NES with different
69 activation ranges for both linear and nonlinear systems. In this way, the efficiency
70 and robustness of TET can be improved, even at small energy levels.

71 About multi NES, there already exist extensive studies [40, 41, 42, 43], and
72 normally frequency components are used as indices to judge the possible tran-
73 sient or sustained resonance captures, and to identify different levels of activation.

74 However, this kind of index will not be used here for VI NES, since the impact
75 number per cycle is a more direct measure. For example, the response regime with
76 two impacts per cycle can be easily distinguished from numerical or experimen-
77 tal viewpoints, and actually it corresponds to 1:1 resonance. Although frequency
78 components can be calculated by wavelet transform, it will not make the analysis
79 easier and will not be used here.

80 In addition to the viewpoint of TET, the analytical study of systems coupled
81 with multi VI NES also deserves further investigation for two reasons. Firstly, the
82 research of multi VI NES is closely related to particle dampers [44]. Secondly,
83 complicated response regimes such as intermittent beating responses [18, 19] need
84 further analytical explanations rather than only analytical treatments of steady
85 periodic regimes [16, 20].

86 Therefore, the objective of this paper is to generalize the optimization design
87 criterion for one VI NES [36] to multiple VI NES. The focus is still around re-
88 sponse regimes, efficiency and the relation between them. The impact number per
89 cycle is used to characterize response regimes and corresponding TET.

90 The paper is organized as follows: a linear oscillator (LO) coupled with two VI
91 NES in parallel under periodic excitation is analytically developed in Section 2.
92 Then, numerical validations are demonstrated. In Section 4, experimental results
93 under periodic and transient excitations are demonstrated for different conditions.
94 Finally, a conclusion is addressed.

95 **2. Analytical development**

96 The considered system is presented in Fig. 1. Two VI NES in parallel are
97 attached to a LO. The equation of motion between impacts is expressed as follows:

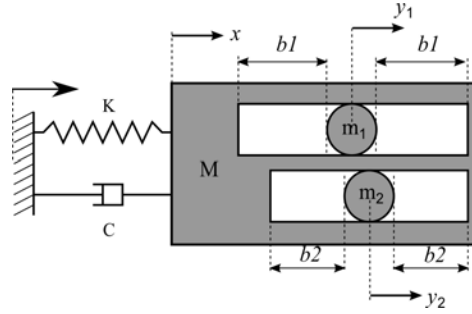


Fig. 1. Schema of a LO with two VI NES coupled in parallel.

$$\begin{aligned}
 M \frac{d^2 x}{dt^2} + C \frac{dx}{dt} + Kx &= F \sin \Omega t \\
 m_1 \frac{d^2 y_1}{dt^2} &= 0, \quad \forall |x - y_1| < b_1 \\
 m_2 \frac{d^2 y_2}{dt^2} &= 0, \quad \forall |x - y_2| < b_2
 \end{aligned} \tag{1}$$

98 The motion relation between states before and after impacts can be obtained
 99 under the condition of simplified impact theory and conservation of momentum
 100 and it can be written as:

$$\begin{aligned}
 &\forall |x - y_1| = b_1 \quad \text{or} \quad |x - y_2| = b_2 \\
 &x_+ = x_-, \quad y_{1+} = y_{1-}, \quad y_{2+} = y_{2-} \\
 &M \frac{dx_+}{dt} + m_1 \frac{dy_{1+}}{dt} + m_2 \frac{dy_{2+}}{dt} = M \frac{dx_-}{dt} + m_1 \frac{dy_{1-}}{dt} + m_2 \frac{dy_{2-}}{dt} \\
 &\text{if } |x - y_1| = b_1 \\
 &\frac{dx_+}{dt} - \frac{dy_{1+}}{dt} = -R \left(\frac{dx_-}{dt} - \frac{dy_{1-}}{dt} \right), \quad \frac{dy_{2+}}{dt} = \frac{dy_{2-}}{dt} \\
 &\text{if } |x - y_2| = b_2 \\
 &\frac{dx_+}{dt} - \frac{dy_{2+}}{dt} = -R \left(\frac{dx_-}{dt} - \frac{dy_{2-}}{dt} \right), \quad \frac{dy_{1+}}{dt} = \frac{dy_{1-}}{dt}
 \end{aligned} \tag{2}$$

101 The subscripts + and - indicate time immediately after and before impacts.

102 R represents the restitution coefficient of impact. The dimensionless variables as
 103 follows are introduced:

$$\begin{aligned} \omega^2 &= \frac{K}{M}, \quad T = \omega t, \quad \lambda = \frac{C}{(m_1 + m_2)\omega}, \quad \varepsilon = \frac{m_1 + m_2}{M}, \\ G &= \frac{F}{(m_1 + m_2)\omega^2}, \quad X = \frac{x}{b_1}, \quad Y_1 = \frac{y_1}{b_1}, \\ Y_2 &= \frac{y_2}{b_1}, \quad \Delta = \frac{b_2}{b_1}, \quad \alpha_i = \frac{m_i}{m_1 + m_2}, \quad \tilde{\Omega} = \frac{\Omega}{\omega} \end{aligned} \quad (3)$$

104 After substitution of Eq.(3) in Eq.(1), it becomes:

$$\begin{aligned} \ddot{x} + \varepsilon\lambda\dot{X} + X &= \varepsilon G \sin \tilde{\Omega} T, \\ \varepsilon\alpha_1\ddot{Y}_1 &= 0, \\ \varepsilon\alpha_2\ddot{Y}_2 &= 0, \quad \forall |X - Y_1| < 1 \text{ and } |X - Y_2| < \Delta \end{aligned} \quad (4)$$

105 In the same way, after substitution of Eq.(3) in Eq.(2), it becomes:

$$\begin{aligned} \forall |X - Y_1| = 1 \quad \text{or} \quad |X - Y_2| = \Delta \\ X_+ = X_-, \quad Y_{1+} = Y_{1-}, \quad Y_{2+} = Y_{2-} \\ M\dot{X}_+ + \varepsilon\alpha_1\dot{Y}_{1+} + \varepsilon\alpha_2\dot{Y}_{2+} &= M\dot{X}_- + \varepsilon\alpha_1\dot{Y}_{1-} + \varepsilon\alpha_2\dot{Y}_{2-} \\ \text{if } |X - Y_1| = 1 \quad \dot{X}_+ - \dot{Y}_{1+} &= -R(\dot{X}_- - \dot{Y}_{1-}), \quad \dot{Y}_{2+} = \dot{Y}_{2-} \\ \text{if } |X - Y_2| = \Delta \quad \dot{X}_+ - \dot{Y}_{2+} &= -R(\dot{X}_- - \dot{Y}_{2-}), \quad \dot{Y}_{1+} = \dot{Y}_{1-} \end{aligned} \quad (5)$$

106 The barycentric coordinates are introduced in the following way:

$$V = X + \varepsilon\alpha_1 Y_1 + \varepsilon\alpha_2 Y_2, \quad W_1 = X - Y_1, \quad W_2 = X - Y_2 \quad (6)$$

107 These coordinates correspond to the physical displacement of the center of
 108 mass and the relative displacement of VI NES. The substitution of Eq.(6) in Eq.(4)
 109 and Eq.(5) gives:

$$\begin{aligned}
& \ddot{V} + \frac{V + \varepsilon\alpha_1 W_1 + \varepsilon\alpha_2 W_2}{1 + \varepsilon} + \varepsilon\lambda \frac{\dot{V} + \varepsilon\alpha_1 \dot{W}_1 + \varepsilon\alpha_2 \dot{W}_2}{1 + \varepsilon} \\
& = \varepsilon G \sin \tilde{\Omega} T \\
& \ddot{W}_1 + \frac{V + \varepsilon\alpha_1 W_1 + \varepsilon\alpha_2 W_2}{1 + \varepsilon} + \varepsilon\lambda \frac{\dot{V} + \varepsilon\alpha_1 \dot{W}_1 + \varepsilon\alpha_2 \dot{W}_2}{1 + \varepsilon} \\
& = \varepsilon G \sin \tilde{\Omega} T \\
& \ddot{W}_2 + \frac{V + \varepsilon\alpha_1 W_1 + \varepsilon\alpha_2 W_2}{1 + \varepsilon} + \varepsilon\lambda \frac{\dot{V} + \varepsilon\alpha_1 \dot{W}_1 + \varepsilon\alpha_2 \dot{W}_2}{1 + \varepsilon} \\
& = \varepsilon G \sin \tilde{\Omega} T
\end{aligned} \tag{7}$$

$$\forall |W_1| < 1 \text{ and } |W_2| < \Delta$$

$$\forall |W_1| = 1 \text{ or } |W_2| = \Delta$$

$$V_+ = V_-, \quad W_{1+} = W_{1-},$$

$$W_{2+} = W_{2-}, \quad \dot{V}_+ = \dot{V}_-$$

$$\text{if } |W_1| = 1$$

$$\dot{W}_{1+} = -R\dot{W}_{1-}, \quad \dot{W}_{2+} = \dot{W}_{2-} - \frac{\varepsilon\alpha_1(1+R)}{1+\varepsilon\alpha_1}\dot{W}_{1-}$$

$$\text{if } |W_2| = \Delta$$

$$\dot{W}_{2+} = -R\dot{W}_{2-}, \quad \dot{W}_{1+} = \dot{W}_{1-} - \frac{\varepsilon\alpha_2(1+R)}{1+\varepsilon\alpha_2}\dot{W}_{2-}$$

110 Eqs.(7) and (8) are analyzed with the multiple scales method. Solutions are
 111 found in the following form:

$$\begin{aligned}
V(T_0, T_1) &= V_0(T_0, T_1) + \varepsilon V_1(T_0, T_1) + \dots \\
W_1(T_0, T_1) &= W_{10}(T_0, T_1) + \varepsilon W_{11}(T_0, T_1) + \dots \\
W_2(T_0, T_1) &= W_{20}(T_0, T_1) + \varepsilon W_{21}(T_0, T_1) + \dots
\end{aligned} \tag{9}$$

112 with $T_i = \varepsilon^i T$. After substitution of Eq.(9) in Eq.(7) and Eq.(8), they become
 113 as follows:

114 • Ordre ε^0

$$\begin{aligned}
& \forall |W_{10}| < 1 \text{ and } |W_{20}| < \Delta \\
& D_0^2 V_0 + V_0 = 0, \\
& D_0^2 W_{10} + V_0 = 0, \quad D_0^2 W_{20} + V_0 = 0 \\
& \forall |W_{10}| < 1 \text{ or } |W_{20}| < \Delta \\
& V_{0+} = V_{0-}, \quad W_{10+} = W_{10-}, \\
& W_{20+} = W_{20-}, \quad D_0 V_{0+} = D_0 V_{0-} \\
& \text{if } |W_{10}| = 1 \\
& D_0 W_{10+} = -R D_0 W_{10-}, \quad D_0 W_{20+} = D_0 W_{20-} \\
& \text{if } |W_{20}| = \Delta \\
& D_0 W_{20+} = -R D_0 W_{20-}, \quad D_0 W_{10+} = D_0 W_{10-}
\end{aligned} \tag{10}$$

115 • Ordre ε^1

$$\begin{aligned}
& D_0^2 V_1 + V_1 = -\alpha_1 W_{10} - \alpha_2 W_{20} \\
& + V_0 - 2D_0 D_1 V_0 - \lambda D_0 V_0 + G \sin \Omega T_0
\end{aligned} \tag{11}$$

116 The solution of the first equation of Eq.(10) can be written in the following
117 way:

$$V_0(T_0, T_1) = A(T_1) \sin(T_0 + \alpha(T_1)) \tag{12}$$

118 The second and third equation of Eq.(10) represent an oscillator with vibro-
119 impacts under periodic excitations. Like in [29], Their solutions can be expressed
120 with standard trigonometric functions as follows:

$$\begin{aligned}
& W_{j0}(T_0, T_1) = A(T_1) \sin(T_0 + \alpha(T_1)) \\
& + \frac{2B_j}{\pi} \arcsin(\cos(T_0 + \eta_j(T_1))), \quad j = 1, 2
\end{aligned} \tag{13}$$

121 Combined with the conditions of impacts given in Eq.(10), the following rela-
 122 tions are obtained:

$$\begin{aligned}\sin(\alpha - \eta_1) &= \frac{1 - B_1}{A}, & \cos(\alpha - \eta_1) &= \frac{2B_1\delta}{A\pi} \\ \sin(\alpha - \eta_2) &= \frac{\Delta - B_2}{A}, & \cos(\alpha - \eta_2) &= \frac{2B_2\delta}{A\pi}\end{aligned}\quad (14)$$

123 with $\delta = (1 - R)/(1 + R)$, these relations can be combined through using the
 124 trigonometric identity:

$$A^2 = (1 - B_1)^2 + \frac{4B_1^2\delta^2}{\pi^2} = (\Delta - B_2)^2 + \frac{4B_2^2\delta^2}{\pi^2}\quad (15)$$

125 Eq.(15) represents one algebraic relation related to the displacement amplitude
 126 of LO and that of two VI NES, which is termed as SIM. In the first approximation
 127 order, the amplitude of every VI NES only depends on the amplitude of the main
 128 system and the performance of two VI NES is decoupled. It establishes the ana-
 129 lytical foundation of the principle of separate activation as stated in [42, 43] for
 130 the case of cubic NES, which is also applicable for VI NES here.

131 Eq.(11) in the order ε^1 is now studied. To identify secular terms, the solutions
 132 of W_{j0} are developed in series of Fourier. The development for one harmonic
 133 gives:

$$W_{j0} = A \sin(T_0 + \alpha) + \frac{8B_j}{\pi^2} \cos(T_0 + \eta_j), \quad j = 1, 2\quad (16)$$

134 The system is studied in the vicinity of the resonance frequency of the main
 135 system. The pulsation of excitation is expressed in the following way:

$$\tilde{\Omega} = 1 + \varepsilon\sigma\quad (17)$$

136 After the substitution of Eqs.(12, 16) and Eq.(17) in Eq.(11), it comes:

$$\begin{aligned}
 & D_0^2 V_1 + V_1 \\
 & = - \sum_{j=1}^2 \left[\alpha_j \left(A \sin(T_0 + \alpha) + \frac{8B_j}{\pi^2} \cos(T_0 + \eta_j) \right) \right] \\
 & - 2D_1 A \cos(T_0 + \alpha) + A \sin(T_0 + \alpha) (1 + 2D_1 \alpha) \\
 & - \lambda A \cos(T_0 + \alpha) + G \sin(T_0 + \sigma T_1)
 \end{aligned} \tag{18}$$

137 After arrangement, the condition for the elimination of secular term is as fol-
138 lows:

$$\begin{aligned}
 2D_1 A & = - \sum_{j=1}^2 \left[\frac{8\alpha_j B_j}{\pi^2} \cos(\alpha - \eta_j) \right] - \lambda A + G \sin(T_1 \sigma - \alpha) \\
 2AD_1 \alpha & = \sum_{j=1}^2 \left[\frac{8\alpha_j B_j}{\pi^2} \sin(\alpha - \eta_j) \right] - G \cos(T_1 \sigma - T_1)
 \end{aligned} \tag{19}$$

139 The expressions of $\sin(\alpha - \eta_j)$ and $\cos(\alpha - \eta_j)$ ($j = 1, 2$) given in Eq.(14) are
140 substituted in Eq.(19), then the change of variable $T_1 \sigma - \alpha = \theta$ is introduced:

$$\begin{aligned}
 2D_1 A & = \frac{16\alpha_1 B_1^2 \delta}{\pi^3 A} - \frac{16\alpha_2 B_2^2 \delta}{\pi^3 A} - \lambda A + G \sin \theta \\
 2AD_1 \theta & = \frac{8\alpha_1 B_1 (1 - B_1)}{A\pi^2} - \frac{8\alpha_2 B_2 (\Delta - B_2)}{A\pi^2} + G \cos \theta + 2A\sigma
 \end{aligned} \tag{20}$$

141 The fixed points to the limit $T_1 \rightarrow \infty$ are calculated by eliminating the deriva-
142 tion in Eq.(20). The obtained equations can be combined by using the trigonomet-
143 ric identity. After rearrangement, one equation of the following form is obtained:

$$c_2 A^4 + c_1(B_1, B_2) A^2 + c_0(B_1, B_2) = 0 \tag{21}$$

144 Eq.(21) can be expressed in the function of only A and B_1 or B_2 by using
145 Eq.(15). The obtained equation can be resolved for A^2 .

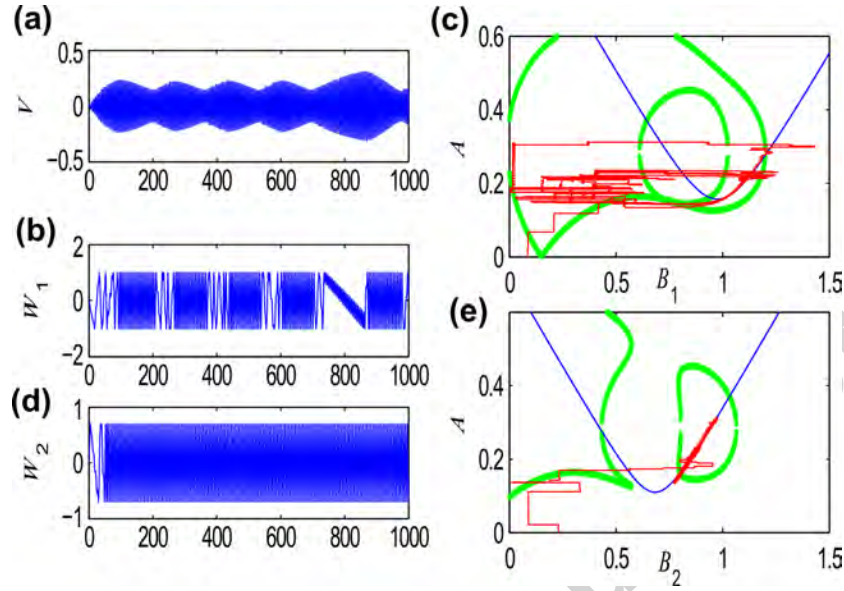


Fig. 2. Numerical integration of Eqs. (7, 8) for $G = 0.8$ and $\sigma = 0.5$: (a) SMR of V ; (b) intermittent resonance of W_1 related to the first VI NES; (c) the analytically obtained SIM of the first order in blue curve and of the second order in green curve, the projection of motion for the first VI NES in red curve; (d) the constant response of W_2 related to the second VI NES; (e) the projection of motion for the second VI NES in red curve.

146 The thereby obtained algebraic relation between A and B_1 or B_2 represents the
 147 invariant manifold of the problem in the time-scale T_1 . Subsequently, different
 148 results of numerical integration and their projections into the invariant manifold
 149 of each VI NES are presented.

150 3. Numerical results

151 In this part, the objective is to validate the analytical results by numerical
 152 simulations. The Eqs.(7) and (8) will be used. The used parameters are showed
 153 as follows:

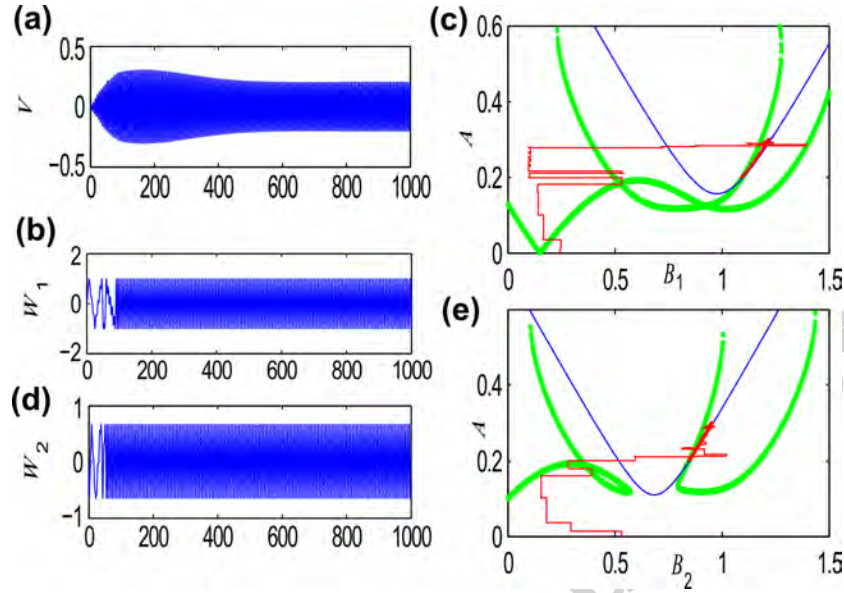


Fig. 3. Numerical integration of Eqs. (7, 8) for $G = 0.9$ and $\sigma = -0.2$: (a) steady state response of V ; (b) constant resonance of W_1 related to the first VI NES; (c) the analytically obtained SIM of the first order in blue curve and of the second order in green curve, the projection of motion for the first VI NES in red curve; (d) the constant response of W_2 related to the second VI NES; (e) the projection of motion for the second VI NES in red curve.

$$\begin{aligned} \varepsilon = 0.01, \quad \Delta = 0.7, \quad R = 0.6, \\ \alpha_1 = \alpha_2 = 0.5, \quad \lambda = 1 \end{aligned} \quad (22)$$

154 The numerical integration results with SIM for $G = 0.8$ and $\sigma = 0.5$ are pre-
 155 sented in Fig. 2. In this case, the time history of V , which is an approximate
 156 description of the displacement of LO, illustrates its strongly modulated feature
 157 [31, 34] as displayed in Fig. 2(a). Its amplitude is chaotic strongly modulated
 158 and the relative maximal amplitude is not constant every time. In Fig. 2(b), the
 159 relative displacement related to VI NES 1 is showed. The dense parts denote that

160 it realizes two impacts per cycle, and the sparse parts denote the occasional out of
 161 activation. Therefore, VI NES1 realizes intermittent response . Then, its response
 162 is projected to the SIM in red curve as showed in Fig. 2(c). The intersections
 163 of the blue curve and the green curve denote the fixed points, but their stabilities
 164 are not calculated here. The overlapping parts between the blue curve and the
 165 red curve demonstrate the parts with two impacts per cycle and validate the prediction
 166 accuracy of SIM. For VI NES 2, it is activated in permanence with two
 167 impacts per cycle as showed by W_2 in Fig. 2(d). Its motion is projected to SIM
 168 as showed in Fig. 2(e), and the projected A and B_2 are always in right branch
 169 of SIM. It means that the displacement amplitude of VI NES 2 will always vary
 170 simultaneously with that of LO.

171 Another case is presented in Fig. 3 for $G = 0.9$ and $\sigma = -0.2$. In this case,
 172 the two VI NES are attracted by fixed points, namely the intersections between
 173 the blue curve and red curve, as showed in Fig. 3 (c) and (e). The main system
 174 performs steady oscillation as displayed in Fig. 3 (a). Compared to the last case,
 175 the values of A , B_1 and B_2 are constant and well predicted by the SIMs.

176 Actually, the ability of SIM obtained by analytical study is limited on predict-
 177 ing the resonance responses with two impacts per cycle of main system or other
 178 response regimes with resonant parts. Other more complicated phenomena will
 179 be demonstrated by the following experimental study.

180 4. Experimental results

181 In this section, experiments are done for periodic and transient excitation re-
 182 spectively to compare the case with one VI NES and that with two VI NES. For pe-
 183 riodic excitation, the experimentally obtained and used parameters are displayed

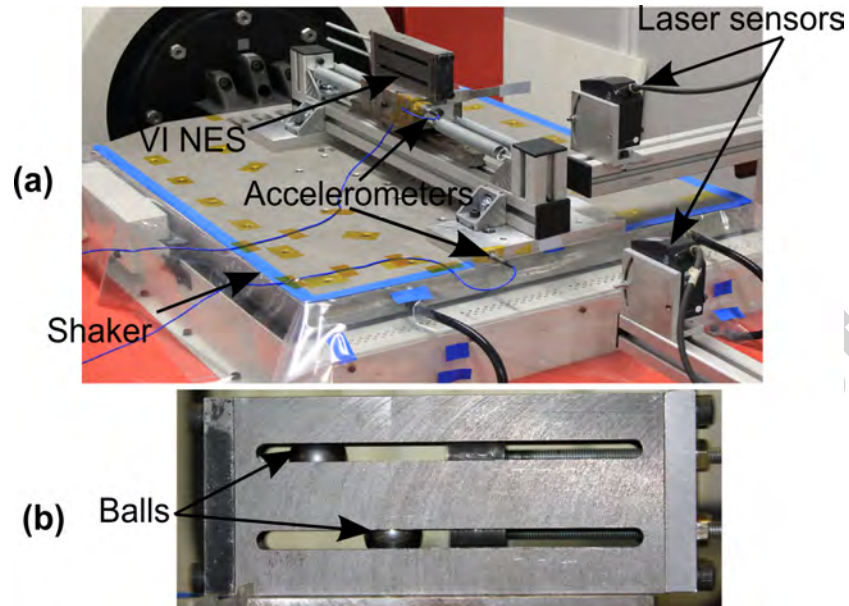


Fig. 4. Experimental setup: (a) global configuration; (b) detailed view of VI NES.

184 at the first place. Then, the results under excitation with a single frequency and
 185 a range of frequency around resonance frequency are demonstrated to show the
 186 possible response regimes, and to compare the efficiency under different combi-
 187 nations of length of cavity. Regarding transient excitation, the same experimental
 188 device is applied and the objective is to verify the principle of separate activation.

189 4.1. Periodic excitation

190 4.1.1. Experimental setup

191 The global experimental configuration is showed in Fig. 4(a). Two VI NES
 192 are put inside two clearances of LO in parallel as demonstrated in Fig. 4 (b), and
 193 they can move freely inside. The whole system is embedded on an electrodynamic
 194 shaker with a maximal force 10 kN. The displacement of LO as well as the im-

Table 1. Experimental parameters

Physical Parameters			
M	4.7 kg	C	3.02 Ns/m
K	$11.47 \cdot 10^3$ N/m		
m_1	32 g	m_2	32 g
b_1	0 – 50 mm	b_2	0 – 50 mm
Reduced Parameters			
ε	0.76 %	λ_1	1.91
f_0	7.86 Hz		
Single frequency test			
f_r	7.82 – 7.84 Hz		
Shaker acceleration	0.06 g		
Frequency band test			
$f_s - f_e$	6.5 – 9 Hz		
Shaker acceleration	0.06 g		

195 posed displacement of the shaker are measured by contact-less laser displacement
196 sensors. Their accelerations are measured by accelerometers and the impacts be-
197 tween VI NES and LO can be judged from the sudden changes of the acceleration
198 of LO. A detailed view of the configuration for two VI NES is presented in Fig. 4
199 (b). It simply consists of two closed clearances of lengths $d + 2 \cdot b_1$ and $d + 2 \cdot b_2$,
200 respectively, where d is the diameter of both balls (VI NES). b_1 and b_2 are lengths
201 of the above clearance and the below clearance, respectively, and each can be ad-
202 justed by a cylinder. The cylinder and the cover at the opposite side are made of
203 hardened steel. The parameters of this system have been identified by performing
204 modal analysis and are summarized in Table 1.

205 4.1.2. Single frequency excitation

206 The frequency of the sinusoidal excitation is slowly varied from 7.82 Hz to
207 7.84 Hz during 80 s, which can be considered almost fixed to the value 7.83 Hz.
208 This value is closed to the natural frequency of LO ($f_0 = 7.86$ Hz). The acceler-
209 ation of the shaker is fixed to 0.06 g. During the whole experimental process, the
210 time histories of displacement and acceleration of LO are recorded.

211 With the change of the number of VI NES and different combinations of clear-
212 ances b_1 and b_2 , different periodic and transient response regimes are observed
213 and demonstrated here. They are identified by the difference of impact numbers
214 per cycle of LO, which can be judged from the time history of the acceleration of
215 LO.

216 At first, the time history of the acceleration of LO without VI NES is showed in
217 Fig. 5(a) as a reference. Although there exist small shakes at some of its maximal
218 place, no impacts exist. In addition, its amplitude is highest compared to other
219 cases coupled with VI NES. Then, the time history of the acceleration of LO with

220 a VIBNES for a clearance of 30 mm is showed in Fig. 5(b) as another reference.
221 This value of clearance is proved to be almost optimal in the sense of vibration
222 control and the response regime is two impacts per cycle [36]. It is not so easy
223 to distinguish impacts because the order of impact strength is close to that of the
224 acceleration of LO, but the impact number is observable during the process of
225 experiment. Moreover, the peaks of acceleration are evidently reduced compared
226 to the results in Fig. 5(a).

227 With the addition of another ball and different combinations of clearances, the
228 response regimes will be more complicated. However, the possible variations are
229 always based on the above two cases. Then, the regime with four impacts per cycle
230 is observed with the addition of a second ball with clearance $b_1 = 5$ mm (another
231 ball with clearance $b_2 = 30$ mm), and both ball impact twice per cycle as showed
232 in Fig. 5(c), and the four impact moments during one cycle are marked out in
233 rectangles. The acceleration of LO is further decreased compared to the optimized
234 one ball case with 30 mm clearance, and this efficiency improvement cannot be
235 easily judged here and will be demonstrated from the viewpoint of displacement
236 later.

237 When the clearance of the added ball is increased to 40 mm, the response is
238 complicated as displayed in Fig. 5(d). The ball below with $b_2 = 30$ mm con-
239 tinuously impacts twice per cycle. In contrast, the added ball impacts only once
240 during many cycles (> 1), and a few of them are marked out in rectangles.

241 In addition to the above relative stable response regimes, there exist some
242 even more complicated transient response regimes during one time history for
243 some combinations of clearances. For the case with $b_1 = b_2 = 30$ mm, there are
244 some periods as showed in Fig. 5(e), in which just one ball is activated with two

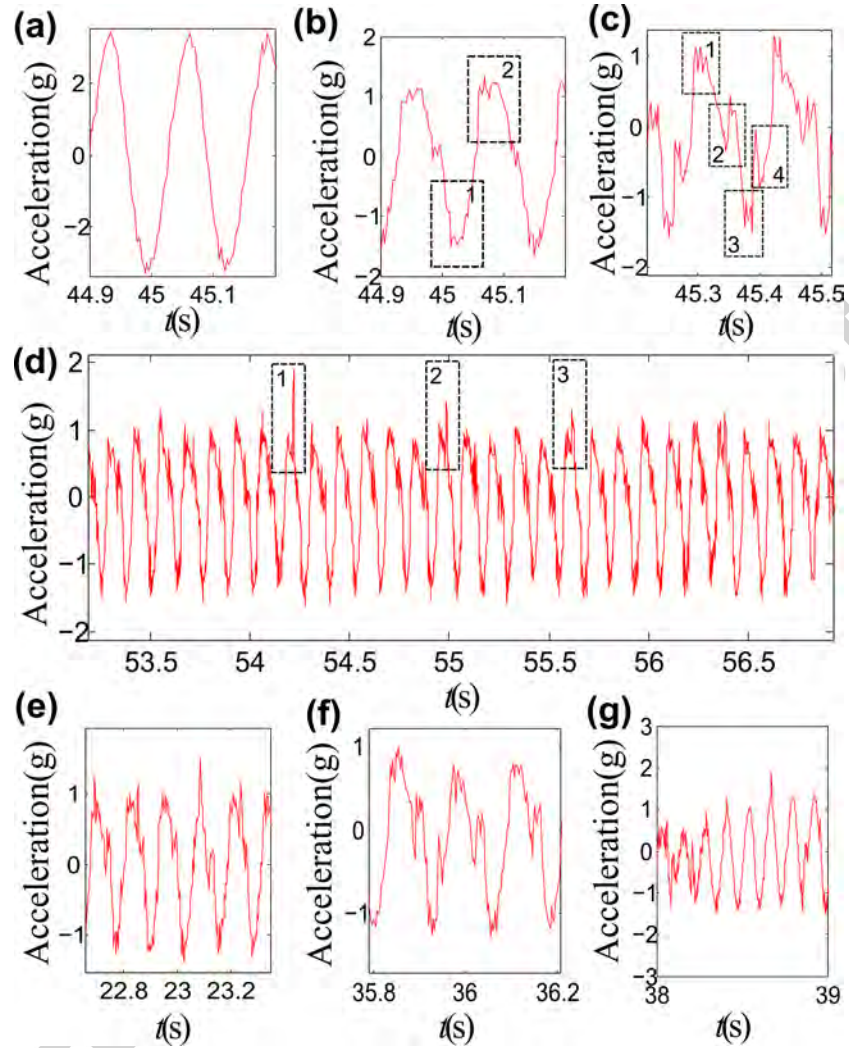


Fig. 5. Typical response regimes by comparison of impact numbers per cycle of the acceleration of LO and some impacts moments are marked out in rectangle: (a) no impact and no VI NES; (b) two impacts per cycle; (c) four impacts per cycle; (d) one VI NES impacts once during many cycles and another VI NES impacts twice per cycle; (e) one period of a response with at least one VI NES in the state of two impacts per cycle; (f) one period of a response with both VI NES in the state of two impacts per cycle; (g) one period of a response with both VI NES out of activation and no impact.

245 impacts per cycle, but this activated state can alternate between these two balls.
246 This phenomenon is complicated, and it cannot be judged from the results showed
247 here and can just be observed in the test site. There are also some periods, two
248 balls impact twice per cycle as showed in Fig. 5(f) and there are even some periods
249 there are almost no impacts as showed in Fig. 5(g).

250 Therefore, the strong nonlinear coupling between these two balls and LO is
251 well observed by the complicated variations of response regimes during one time
252 history. The above mentioned basic response regimes will be applied to explain
253 the complication variation of efficiency, and this point is closely related to the
254 targeted energy transfer by transient resonance captures.

255 In [36], it is observed that the optimal response regime is the one with two im-
256 pacts per cycle and around the entrance of SMR. The idea of optimization design
257 is to make efficient response regimes occur for different excitations. This idea can
258 still apply in the optimization design of two VI NES, namely to make each VI
259 NES activated at its best state with two impacts per cycle.

260 Then, the efficiency comparison of different combinations of clearances is per-
261 formed here to observe the possible relation between the types of response regimes
262 and their efficiency. The two cases without VI NES and with one optimized VI
263 NES (30 mm clearance) are chosen as two references as showed in Fig. 6(a-b).

264 The length of the down clearance b_2 is fixed to 30 mm and only the upper
265 clearance b_1 is varied from 5 mm to 50 mm. Here, only the time histories of
266 displacement are demonstrated for $b_1 = 30$ mm and $b_1 = 5$ mm, respectively.
267 The former is displayed in Fig. 6(a), there are three typical areas A1, A2 and
268 A3, which correspond to typical response regimes from Fig. 5(e) to Fig. 5(g),
269 respectively. In area A1, there is a small decrease of amplitude compared to the

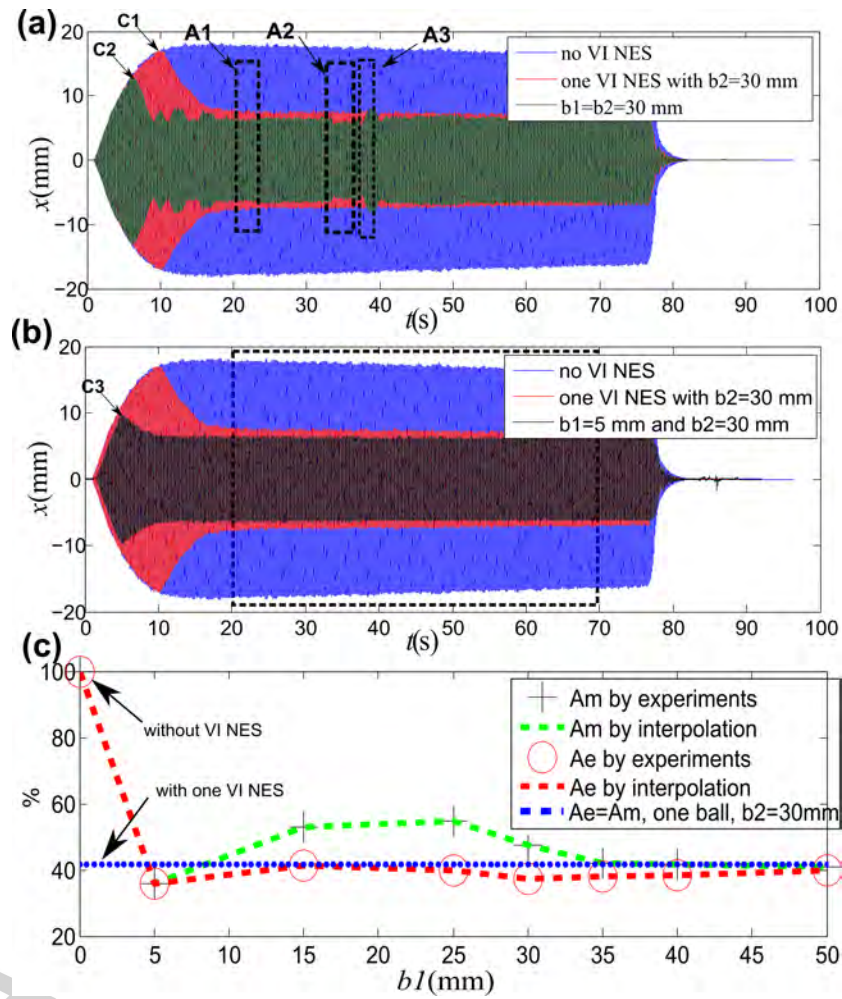


Fig. 6. Typical response regimes and their efficiency comparison with different combinations of clearances: (a-b) response regimes; (c) efficiency comparison by average and maximal amplitude ratios (A_e and A_m).

270 one VI NES case in red curve. In area A2, both ball impact twice per cycle and the
271 amplitude is lowest. In area A3, the occasional out of activation for both balls and
272 the amplitude increases. In the whole process, there are many transitions between
273 them, which results in the complicated variation of displacement amplitude of LO.
274 It is a direct proof of nonlinear coupling between two VI NES and LO. In addition,
275 the efficiency is highest when both balls impact twice per cycle, which is the most
276 effective form of transient resonance captures.

277 This efficient targeted energy transfer is better demonstrated by the decrease
278 of b_1 to 5 mm as showed in Fig. 6(b). This time, both VI NES impact twice per
279 cycle and it means the permanent resonance captures.

280 With the excitation of LO at the beginning, different activation amplitudes of
281 LO with different combinations of clearances are observed, namely C1, C2 and C3
282 in Fig. 6(a-b). With the increase of VI NES from 0 to 1, and then 2, the amplitude
283 is evidently decreased at the activation point of VI NES. The same conclusion
284 is obtained when the value of another clearance is decreased. In this sense, the
285 robustness and efficiency can be improved.

286 Then, the average and maximal amplitude ratios between the case with VI
287 NES and without NES are calculated for all cases during a stable time period (20-
288 70 s), and the results are showed in Fig. 6(c). Compared to the case without VI
289 NES and with one VI NES, the optimal case is the addition of another VI NES
290 with a small clearance 5 mm. For other cases with large clearances, it cannot im-
291 prove the efficiency and, in return, it will result in the occasional out of activation
292 of VI NES, and meanwhile, the large displacement of LO.

293 According to the optimization design criterion proposed in [36], the design
294 of parameters is to make the efficient response regimes appear and it means the

295 most efficient transfer and dissipation of energy. From the above experimental
296 results with limited combinations of clearances, this optimization design criterion
297 still applies since both balls impacts twice in the optimal case with $b_1 = 5$ mm.
298 However, the phase difference of two VI NES cannot be measured or calculated,
299 it may be another factor related to efficiency.

300 Except the above combinations of clearances, experiments with the same clear-
301 ance for both VI NES are also performed. The average and maximal amplitude
302 ratios are also calculated and showed in Fig. 7. From the viewpoint of A_e , the case
303 with $b_1 = b_2 = 30$ mm is optimal. On the contrary, the case with $b_1 = b_2 = 15$
304 mm is optimal in terms of A_m . Generally speaking, they are less effective than
305 the combination with $b_1 = 5$ mm and $b_2 = 30$ mm.

306 As a summary, the objective to find the relation between response regimes
307 and their efficiency is accomplished in two steps. The first step is to find basic
308 response regimes. Then, they are applied to explain the complicated variation of
309 response. As a result, a general design criterion can also be proposed here: the
310 design of two VI NES is to make them activated around most efficient response
311 regimes.

312 4.1.3. Excitation with a band of frequency

313 Then, the band of excitation frequency is enlarged around the natural fre-
314 quency of LO. The objective here is to study the influence of different combi-
315 nations of clearances on the response regimes and their efficiency. The starting
316 frequency f_s and ending frequency f_e during this sweep is showed in Table 1 and
317 the acceleration is still fixed to 0.06 g.

318 For the above experimental configuration, the one VI NES case with a clear-
319 ance 27.5 mm has been observed optimal [36]. Here the clearance of an added

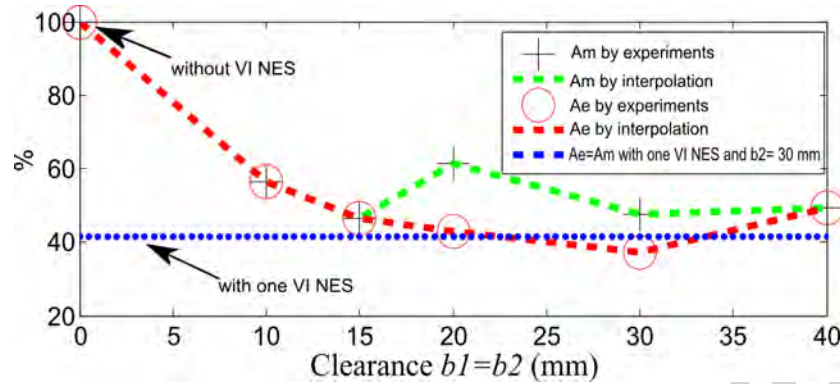


Fig. 7. Efficiency comparison of different response regimes with the same clearance for both VI NES.

320 VI NES is selected around this value. The displacement of LO is recorded for
 321 different combinations of b_1 and b_2 and is showed in Fig. 8. The results with
 322 fixed $b_2 = 27.5$ mm and varying b_1 are showed in Fig. 8(a). The combination of
 323 $b_1 = 12.5$ mm and $b_2 = 27.5$ mm is generally more optimal in this case. Then,
 324 the results with equal b_1 and b_2 are showed in Fig. 8(b). For frequencies a little
 325 below the resonance frequency, the combination of $b_1 = 27.5$ mm and $b_2 = 27.5$
 326 mm is better, but the other two are better for frequencies a little higher than the
 327 resonance frequency. To have a close view, the results between two relative op-
 328 timal cases are compared in Fig. 8(c). In area A, it is observed that VI NES can
 329 be activated at a lower value of displacement amplitude for a smaller b_1 . In area
 330 B, two VI NES are in regime with two impacts per cycle for $b_1 = 27.5$ mm. On
 331 the contrary, one ball occasionally gets out of activation for $b_1 = 12.5$ mm and
 332 results in the transient build-up of amplitude. In area C, low $b_1 = 12.5$ mm still
 333 can be activated for lower amplitude and results in the reduction of amplitude. As
 334 has been demonstrated under the excitation with a fixed frequency, the variation

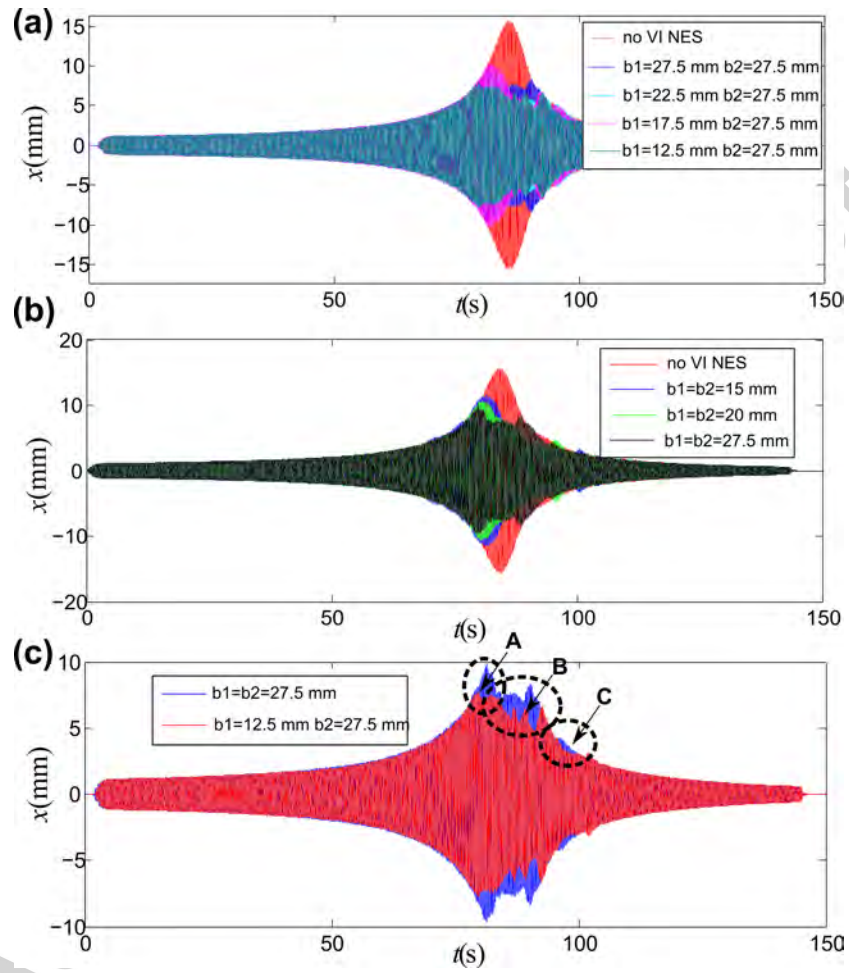


Fig. 8. Responses of LO during the sweep experiments and efficiency comparison: (a) with different b_1 and b_2 ; (b) with the same b_1 and b_2 ; (c) the efficiency comparison for two relative optimal cases.

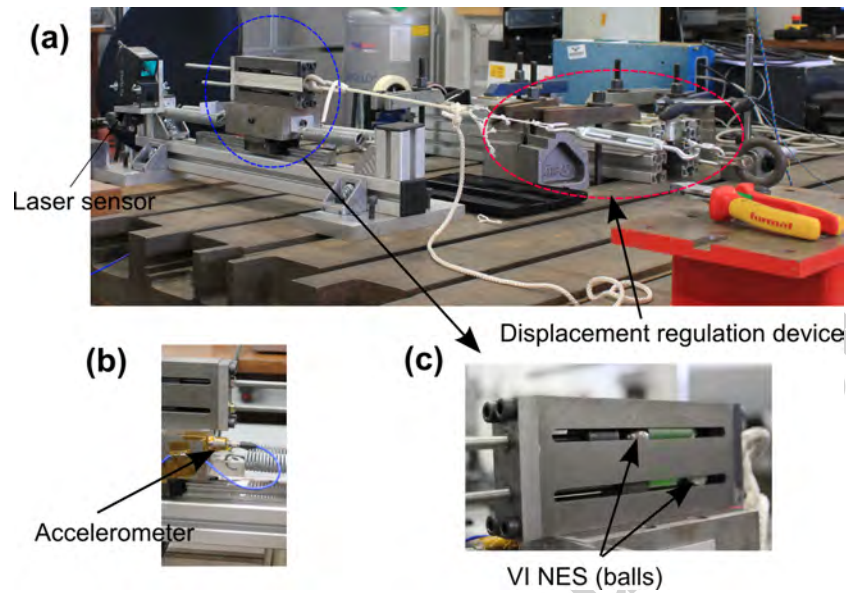


Fig. 9. Experimental setup: (a) global configuration; (b) installation of accelerometer; (c) detailed view of VI NES.

335 of motion for LO and both VI NES can be more complicated than the above-
 336 mentioned characteristics. Except the observed relation between efficiency and
 337 the response regimes, it is desirable to obtain further information but this kind of
 338 try would be difficult.

339 Therefore, the following optimization design criterion is recommended. If
 340 just one ball is applied, it is recommended that VI NES should be optimized at the
 341 point of natural frequency. If two VI NES are applied to improve the robustness
 342 and increase efficiency, a smaller length of clearance should be chosen compared
 343 to the optimized clearance of the one ball case to avoid the occasional failure.

344 4.2. Transient excitation

345 4.2.1. Experimental setup

346 The same experimental device as the periodic case is used, but it is attached to
347 a cast iron bench as showed in Fig. 9(a). The addition of a small ring bolt for pre-
348 stretch will not influence the total mass of LO and its influence can be neglected.
349 Therefore, its parameters are nearly the same as these in Table 1. One laser sen-
350 sor and one accelerometer are used to measure the displacement and acceleration
351 of LO, respectively. The fixation of accelerometer is showed in Fig. 9(b) and a
352 detailed view of VI NES is displayed in Fig. 9(c). The number of balls can be
353 changed. The initial displacement of LO is regulated by a device and is fixed to
354 20 mm for all tests. The initial location of two balls are at random, and the ve-
355 locities for LO and both balls are zero. Since only the stable transition process is
356 studied and the transient process quickly disappears, the initial conditions will not
357 influence the expected conclusion.

358 4.2.2. Principle of separate activation

359 With two sets of b_1 and b_2 , the responses of two cases are compared here. For
360 the case with one VI NES ($b_2 = 20$ mm), the time history of displacement of LO
361 is represented by the red curve in Fig. 10(a) and its corresponding acceleration is
362 showed in Fig. 10(b). The points A1 and A2 are related to the transition from the
363 regime with two impacts per cycle to that without any continuous period of two
364 impacts per cycle.

365 For the case with two balls ($b_1 = 2$ mm and $b_2 = 20$ mm), the time history
366 of displacement of LO is represented by the blue curve in Fig. 10(a) and its cor-
367 responding acceleration is showed in Fig. 10(c). The first VI NES with $b_2 = 20$
368 mm gets out of two impacts per cycle around points B1 and B2. The second VI

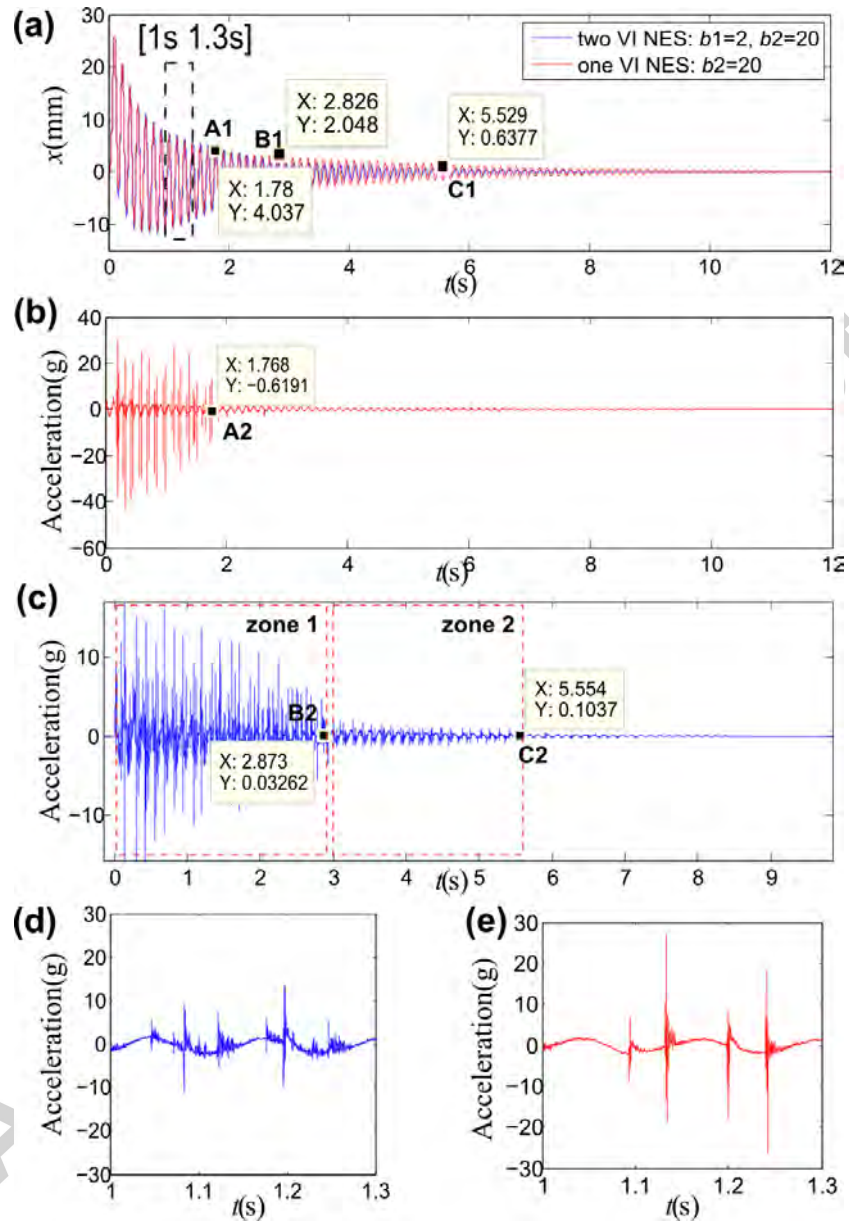


Fig. 10. Response comparison between LO coupled with one VI NES and two VI NES: (a) an imposed time history of displacement; (b) a time history of acceleration with one VI NES; (c) a time history of acceleration with two VI NES; (d) a detailed view of one period of acceleration with two VI NES; (e) a detailed view of one period of acceleration with one VI NES.

369 NES with a small clearance $b_1 = 2$ mm gets out of two impacts per cycle around
370 points C1 and C2. In zone 1, the first VI NES with a large clearance is activated
371 with two impacts per cycle. In zone 2, the first VI NES gets out of activation and
372 the second VI NES with a small clearance is activated with two impacts per cycle.
373 In the sense of the activation with two impacts per cycle, the principle of separate
374 activation of VI NES with different clearances is observed here. This definition
375 of activation is important, since the regime with two impacts per cycle is most
376 efficient in vibration control [36].

377 Then, the decay rates of displacements for both cases are compared. Before
378 point A1, there is almost no difference, since the first VI NES with large clearance
379 is activated with two impacts per cycle and the second VI NES behaves in low
380 efficient regime (more than two impacts per cycle). Between A1 and B1, the role
381 of the second VI NES increases but still small. In contrast, it plays an important
382 role between B1 and C1 since it is activated with two impacts per cycle and the
383 first VI NES is totally out of excitation. The difference of decay rates in this
384 period is relative large. Here, the role of separate activation in vibration control
385 is evident, and the two VI NES with difference clearances can be effective at
386 different ranges of displacement amplitude. More importantly, this effectiveness
387 is related to the efficient response regime with two impacts per cycle.

388 During the above transition process, the first VI NES changes from two im-
389 pacts per cycle to the state with no continuous periods of two impacts per cycle
390 (i.e., permanent out of activation), the limit point is A1. It means that it plays the
391 main role in vibration control before A1. In contrast, the second VI NES changes
392 from more than two impacts per cycle to two impacts per cycle, and finally out of
393 activation, it plays a main role in vibration control between B1 and C1. Between

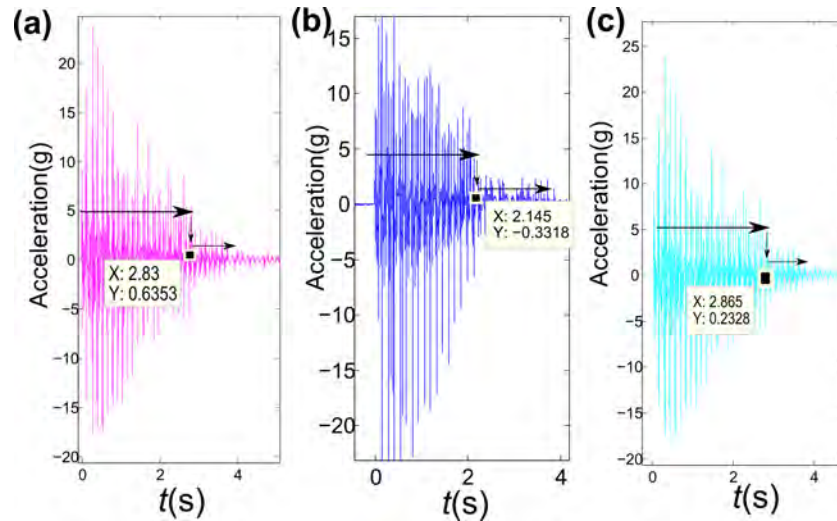


Fig. 11. Principle of separate activation: (a) $b_1 = 5$ mm and $b_2 = 20$ mm; (b) $b_1 = 5$ mm and $b_2 = 30$ mm; (c) $b_1 = 5$ mm and $b_2 = 40$ mm.

394 A1 and B1, it is the overlapping period. Evidently, the second VI NES undergoes
 395 more response regimes and could possess four impacts per cycle, three impacts
 396 per cycle and two impacts per cycle etc., which are closely related to different
 397 transient resonance captures.

398 In addition to displacement decay rates, impacts reflected by acceleration can
 399 also reveal some important characteristics. The time histories of acceleration of
 400 LO from 1 s to 1.3 s are taken out for both cases, as showed in Fig. 10(d) for
 401 two VI NES case and in Fig. 10(e) for one VI NES case. With the addition of
 402 another ball, the impact strengths are decreased and the impacts are scattered. As
 403 observed by other researchers, the regime with strong impact strengths but less
 404 impact numbers is replaced by that with weak and more impacts, but the whole
 405 energy reduction rate is not improved.

406 The principle of separate activation is also observed from results with other

407 combinations of clearances as showed in Fig. 11. The separate points for the
408 out of activation of the VI NES with a large clearance are marked out in each
409 subfigure. The horizontal arrows demonstrate the two activation levels for two
410 different clearances, and the vertical arrows denote the change between them.

411 In summary, the principle of separate activation of VI NES is observed, and
412 the effectiveness of the vibration control can be increased to a large range of dis-
413 placement amplitude of LO. Moreover, the addition of VI NES can reduce impact
414 strengths and scatter impacts though it may not increase the efficiency of vibration
415 control.

416 5. Conclusion

417 A LO coupled with two VI NES in parallel is studied under periodic and tran-
418 sient excitations. Firstly, the system is analytically studied with the multiple scales
419 method. SIM is obtained and fixed points can be calculated. Then, numerical sim-
420 ulations are performed to observe the typical response regimes and to validate the
421 analytical results. Finally, experiments under different types of excitations are car-
422 ried out to observe the possible response regimes and to compare the efficiency of
423 different combinations of clearances.

424 The asymptotic method, which has been proved feasible for the analytical
425 study of systems coupled with one VI NES, is generalized to the case with two VI
426 NES. Two different SIMs are obtained for two VI NES with different clearances.
427 Since a SIM is related to an activation threshold, namely displacement amplitude
428 of LO, the analytically obtained two different SIMs can explain the principle of
429 separate activation. This principle can be more directly reflected by the transient
430 excitation case. In addition to SIM, fixed points can be calculated and their po-

431 sition can be applied to judge the types of response regimes. Compared to the
432 former analytical methods, the variation of transient and modulated response can
433 be further explained this time. Numerical results prove the consistency of ana-
434 lytical prediction of two classic response regimes, namely two impacts per cycle
435 and SMR. More specifically about SMR, its existence and its periods with two
436 impacts per cycle can be well predicted by SIM.

437 Although analytical study is limited to obtain results for regimes with two
438 impacts per cycle and SMR, or any combinations of these two regimes, they are
439 still important. From the viewpoint of TET by resonance captures, these regimes
440 possess the most efficient permanent or transient resonance captures.

441 Through experiments under different excitations, different response regimes
442 are observed and their relation to their efficiency is analyzed. Whatever types
443 of excitation, it is more effective as long as the two VINES are activated with
444 two impacts per cycle or around this regime. In addition to this common law, the
445 results are complicated from the viewpoint of each excitation type. This observa-
446 tion proves the relevance and effectiveness of analytical study around SIM, more
447 precisely, around the response regime with two impacts per cycle (i.e., resonance
448 captures).

449 More specifically about experimental results, several points should be pointed
450 out. Firstly, different basic periodic and transient response regimes are observed
451 and analyzed from the experimental results under excitation with a fixed fre-
452 quency. In addition, more complicated responses are observed but they can be
453 explained by these basic response regimes. Then, they are applied to explain the
454 complicated variation of response regimes, not only for excitation with a fixed fre-
455 quency and a range frequency, but also for transient excitation. All observations

456 here prove the important role of the regime with two impacts per cycle. Under
457 transient excitation, this role is further demonstrated by the separate activation
458 of VI NES of different clearances. This separate activation of VI NES with two
459 impacts per cycle can be applied to control vibration at different levels of displace-
460 ment amplitude. Sometimes, although the addition of VI NES cannot improve the
461 efficiency of vibration control, the impacts are scattered and their strengths are
462 reduced.

463 In a word, many aspects of the dynamics are further explained by the analyt-
464 ical, numerical and experimental studies, but other factors, such as the friction
465 between LO and VI NES, inclinations of clearances and impact models, should
466 be considered in the analytical treatment to improve the accuracy of explanation.

467 **Acknowledgments**

468 The authors acknowledge the French Ministry of Science and the Chinese
469 Scholarship Council [grant number: 201304490063] for their financial support.

470 [1] D. M. McFarland, L. A. Bergman, A. F. Vakakis, Experimental study of non-
471 linear energy pumping occurring at a single fast frequency, *Int. J. Non-Linear*
472 *Mech.* 40 (6) (2005) 891–899.

473 [2] E. Gourc, G. Michon, S. Seguy, A. Berlioz, Experimental investigation and
474 design optimization of targeted energy transfer under periodic forcing, *J.*
475 *Vib. Acoust.* 136 (2) (2014) 021–021.

476 [3] R. Bellet, B. Cochelin, P. Herzog, P.-O. Mattei, Experimental study of tar-
477 geted energy transfer from an acoustic system to a nonlinear membrane ab-
478 sorber, *J. Sound Vib.* 329 (14) (2010) 2768–2791.

- 479 [4] P.-O. Mattei, R. Ponçot, M. Pachebat, R. Côte, Nonlinear targeted energy
480 transfer of two coupled cantilever beams coupled to a bistable light attach-
481 ment, *J. Sound Vib.* 373 (2016) 29–51.
- 482 [5] S. Benacchio, A. Malher, J. Boisson, C. Touzé, Design of a magnetic vibra-
483 tion absorber with tunable stiffnesses, *Nonlinear Dyn.* (2016) 1–19.
- 484 [6] C.-H. Lamarque, O. V. Gendelman, A. T. Savadkoochi, E. Etcheverria, Tar-
485 geted energy transfer in mechanical systems by means of non-smooth non-
486 linear energy sink, *Acta Mech.* 221 (1-2) (2011) 175–200.
- 487 [7] P. Lieber, D. Jensen, An acceleration damper: development, design and some
488 applications, *Trans. ASME* 67 (10) (1945) 523–530.
- 489 [8] R. A. Ibrahim, *Vibro-impact dynamics: modeling, mapping and applica-*
490 *tions*, Vol. 43, Springer Science & Business Media, 2009.
- 491 [9] V. I. Babitsky, *Theory of vibro-impact systems and applications*, Springer
492 Science & Business Media, 2013.
- 493 [10] S. Masri, T. Caughey, On the stability of the impact damper, *J. Appl. Mech.*
494 33 (3) (1966) 586–592.
- 495 [11] C. Bapat, N. Popplewell, K. McLachlan, Stable periodic motions of an
496 impact-pair, *J. Sound Vib.* 87 (1) (1983) 19–40.
- 497 [12] N. Popplewell, C. Bapat, K. McLachlan, Stable periodic vibroimpacts of an
498 oscillator, *J. Sound Vib.* 87 (1) (1983) 41–59.
- 499 [13] C. Bapat, S. Sankar, Single unit impact damper in free and forced vibration,
500 *J. Sound Vib.* 99 (1) (1985) 85–94.

- 501 [14] F. Peterka, More detail view on the dynamics of the impact damper, Me-
502 chanics, Automatic Control and Robotics 3 (14) (2003) 907–920.
- 503 [15] S. Masri, Effectiveness of two-particle impact dampers, J. Acoust. Soc. Am
504 41 (6) (1967) 1553–1554.
- 505 [16] S. Masri, Analytical and experimental studies of multiple-unit impact
506 dampers, J. Acoust. Soc. Am 45 (5) (1969) 1111–1117.
- 507 [17] C. Cempel, The multi-unit impact damper: equivalent continuous force ap-
508 proach, J. Sound Vib. 34 (2) (1974) 199–209.
- 509 [18] C. Bapat, S. Sankar, N. Popplewell, Experimental investigation of control-
510 ling vibrations using multi-unit impact damper, Shock Vib Bull 4 (1983)
511 1–12.
- 512 [19] N. Popplewell, Y. Muzyka, C. Bapat, K. McLachlan, Stable periodic motion
513 of multiple-unit impacting mechanisms, J. Sound Vib. 86 (4) (1983) 587–
514 593.
- 515 [20] C. Bapat, S. Sankar, Multiunit impact damper- re-examined, J. Sound Vib.
516 103 (4) (1985) 457–469.
- 517 [21] Y. Lee, A. F. Vakakis, L. Bergman, D. McFarland, G. Kerschen, F. Nucera,
518 S. Tsakirtzis, P. Panagopoulos, Passive non-linear targeted energy transfer
519 and its applications to vibration absorption: a review, P. I. Mech Eng. K-J
520 Mul. 222 (2) (2008) 77–134.
- 521 [22] A. F. Vakakis, O. Gendelman, L. Bergman, D. McFarland, G. Kerschen,

- 522 Y. Lee, Nonlinear targeted energy transfer in mechanical and structural sys-
523 tems, Vol. 156, Springer Science & Business Media, 2008.
- 524 [23] F. Nucera, A. Vakakis, D. McFarland, L. Bergman, G. Kerschen, Targeted
525 energy transfers in vibro-impact oscillators for seismic mitigation, *Nonlinear
526 Dyn.* 50 (3) (2007) 651–677.
- 527 [24] F. Nucera, F. Lo Iacono, D. McFarland, L. Bergman, A. Vakakis, Application
528 of broadband nonlinear targeted energy transfers for seismic mitigation of a
529 shear frame: Experimental results, *J. Sound Vib.* 313 (1) (2008) 57–76.
- 530 [25] I. Karayannis, A. Vakakis, F. Georgiades, Vibro-impact attachments as shock
531 absorbers, *P. I. Mech. Eng. C-J Mec.* 222 (10) (2008) 1899–1908.
- 532 [26] Y. Lee, F. Nucera, A. Vakakis, D. McFarland, L. Bergman, Periodic orbits,
533 damped transitions and targeted energy transfers in oscillators with vibro-
534 impact attachments, *Physica D* 238 (18) (2009) 1868–1896.
- 535 [27] O. V. Gendelman, Bifurcations of nonlinear normal modes of linear oscilla-
536 tor with strongly nonlinear damped attachment, *Nonlinear Dynamics* 37 (2)
537 (2004) 115–128.
- 538 [28] O. Gendelman, Y. Starosvetsky, M. Feldman, Attractors of harmonically
539 forced linear oscillator with attached nonlinear energy sink i: Description
540 of response regimes, *Nonlinear Dynamics* 51 (1-2) (2008) 31–46.
- 541 [29] O. Gendelman, Analytic treatment of a system with a vibro-impact nonlinear
542 energy sink, *J. Sound Vib.* 331 (21) (2012) 4599–4608.

- 543 [30] E. Gourc, G. Michon, S. Seguy, A. Berlioz, Targeted energy transfer under
544 harmonic forcing with a vibro-impact nonlinear energy sink: Analytical and
545 experimental developments, *J. Vib. Acoust.* 137 (3) (2015) 031008.
- 546 [31] O. Gendelman, A. Alloni, Dynamics of forced system with vibro-impact
547 energy sink, *J. Sound Vib.* 358 (2015) 301–314.
- 548 [32] G. Pennisi, C. Stéphan, G. Michon, Vibro-impact nes: A correlation be-
549 tween experimental investigation and analytical description, in: *Sensors and*
550 *Instrumentation*, Volume 5, Springer, 2016, pp. 137–142.
- 551 [33] T. Li, S. Seguy, A. Berlioz, Dynamics of cubic and vibro-impact nonlin-
552 ear energy sink: Analytical, numerical, and experimental analysis, *J. Vib.*
553 *Acoust.* 138 (3) (2016) 031–010.
- 554 [34] O. Gendelman, A. Alloni, Forced system with vibro-impact energy sink:
555 Chaotic strongly modulated responses, *Procedia IUTAM* 19 (2016) 53–64.
- 556 [35] T. Li, S. Seguy, A. Berlioz, On the dynamics around targeted energy trans-
557 fer for vibro-impact nonlinear energy sink, *Nonlinear Dynamics* (2016) 1–
558 14doi:10.1007/s11071-016-3127-0.
559 URL <http://dx.doi.org/10.1007/s11071-016-3127-0>
- 560 [36] T. Li, S. Seguy, A. Berlioz, Optimization mechanism of targeted energy
561 transfer with vibro-impact energy sink under periodic and transient exci-
562 tation, *Nonlinear Dynamics* (2016) 1–19doi:10.1007/s11071-016-3200-8.
563 URL <http://dx.doi.org/10.1007/s11071-016-3200-8>
- 564 [37] E. Gourc, S. Seguy, G. Michon, A. Berlioz, B. Mann, Quenching chatter

- 565 instability in turning process with a vibro-impact nonlinear energy sink, *J.*
566 *Sound Vib.* 355 (2015) 392–406.
- 567 [38] B. Bergeot, S. Bellizzi, B. Cochelin, Analysis of steady-state response
568 regimes of a helicopter ground resonance model including a non-linear en-
569 ergy sink attachment, *Int. J. Non-Linear Mech.* 78 (2016) 72–89.
- 570 [39] F. Georgiades, A. F. Vakakis, G. Kerschen, Broadband passive targeted en-
571 ergy pumping from a linear dispersive rod to a lightweight essentially non-
572 linear end attachment, *Int. J. Non-Linear Mech.* 42 (5) (2007) 773–788.
- 573 [40] S. Tsakirtzis, G. Kerschen, P. N. Panagopoulos, A. F. Vakakis, Multi-
574 frequency nonlinear energy transfer from linear oscillators to mdof essen-
575 tially nonlinear attachments, *Journal of Sound and Vibration* 285 (1) (2005)
576 483–490.
- 577 [41] S. Tsakirtzis, P. N. Panagopoulos, G. Kerschen, O. Gendelman, A. F.
578 Vakakis, L. A. Bergman, Complex dynamics and targeted energy transfer
579 in linear oscillators coupled to multi-degree-of-freedom essentially nonlin-
580 ear attachments, *Nonlinear Dynamics* 48 (3) (2007) 285–318.
- 581 [42] B. Vaurigaud, A. T. Savadkoohi, C.-H. Lamarque, Targeted energy transfer
582 with parallel nonlinear energy sinks. part i: design theory and numerical
583 results, *Nonlinear Dyn.* 66 (4) (2011) 763–780.
- 584 [43] A. T. Savadkoohi, B. Vaurigaud, C.-H. Lamarque, S. Pernot, Targeted energy
585 transfer with parallel nonlinear energy sinks, part ii: theory and experiments,
586 *Nonlinear Dyn.* 67 (1) (2012) 37–46.

- 587 [44] Z. Lu, X. Lu, S. F. Masri, Studies of the performance of particle dampers
588 under dynamic loads, J. Sound Vib. 329 (26) (2010) 5415–5433.

Accepted manuscript