Linear Parameter Varying systems: from modelling to control

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1. What is a Linear Parameter Varying systems?

2. Modelling and identification of LPV systems

3. Some properties of LPV systems

4. Stability of LPV systems

5. LPV Control & Observation
   - The Dynamic Output feedback case
   - LPV observer design

6. Summary of LPV approach interests
LPV systems

Definition of an Linear Parameter Varying system

\[ \Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \]

\( x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega, \) is a vector of time-varying parameters (\( \Omega \) convex set), assumed to be known \( \forall t \)
What is a Linear Parameter Varying systems?

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(Scherer, ACC Tutorial 2012)

Dampened mass-spring system:

\[ \ddot{p} + c\dot{p} + k(t)p = u, \quad y = x \]

First-order state-space representation:

\[ \frac{d}{dt} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k(t) & -c \end{pmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \]

\[ y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ \dot{p} \end{pmatrix} \]
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\( x(t) \in \mathbb{R}^n \), ...., \( \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in \Omega \), is a vector of time-varying parameters (\( \Omega \) convex set), assumed to be known \( \forall t \)

The frozen Bode plots for \( c = 1 \) and \( k \in [1, 3] \) \((\text{Scherer, ACC Tutorial 2012})\)

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\[ y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{bmatrix} p \\ \dot{p} \end{bmatrix} \]
Let the LPV system be:

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix} \begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\]

\( x(t) \in \mathbb{R}^n, \ldots, \rho = (\rho_1(t), \rho_2(t), \ldots, \rho_N(t)) \in U_\rho, \) is a vector of time-varying parameters \((U_\rho \text{ convex set})\)

- \(\rho(.)\) varies in the set of continously differentiable parameter curves \(\rho : [0, \infty) \rightarrow \mathbb{R}^N\).
  It is assumed to be known or measurable.
- The parameters \(\rho\) are always assumed to be bounded:
  \[
  \rho \in U_\rho \subset \mathbb{R}^N \text{ and } U_\rho \text{ compact} \tag{1}
  \]
  defined by the minimal \(\rho_i,\) and maximal \(\overline{\rho}_i\) values of \(\rho_i(t)\)
  \[
  \rho_i(t) \in [\rho_i, \overline{\rho}_i], \ \forall i
  \]
- The system matrices \(A(.)\) \ldots are continuous on \(U_\rho\)
LPV systems (3): about the parameters

- Parameters are **exogenous** if they are external variables. The system is in that case *non stationary*. See the previous damped mass-spring system.

- Parameters are **endogenous** if they are function of the state variables, \( \rho = \rho(x(t), t) \), and, in that case, the LPV system is referred to as a **quasi-LPV system**. This case is encountered when approximating Nonlinear systems. For instance:

\[
\dot{x}(t) = x^2(t) = \rho(t)x(t)
\]

with \( \rho(t) = x(t) \).

- It is sometimes required that the derivative of the parameters are bounded, i.e:

\[
\dot{\rho} \in U_\dot{\rho} \subset \mathbb{R}^N \text{ and } U_\dot{\rho} \text{ compact}
\]

defined by the minimal \( v_i \), and maximal \( \bar{v}_i \) values of \( \dot{\rho}_i(t) \)

\[
\dot{\rho}_i(t) \in [v_i, \bar{v}_i], \ \forall i
\]

This corresponds to the case of *slow varying parameters*

- Other representations can be considered if \( \rho \) is piecewise-constant, or varies in a finite set of elements (\( \rho(t) \in \{0, 1\} \) for switching systems)

Next, several classes of LPV models are presented, and some ways to go from one class to another are given.
Some comments

- LPV systems can model uncertain systems ($\rho$ fixed but unknown) or parameter-varying models ($\rho(t)$)

LPV=linear or nonlinear?

- What is often referred to as gain-scheduling control, corresponds to Jacobian linearization of the nonlinear plant about a family of equilibrium points Shamma (90), Rugh & Shamma (2000)
  In terms of control design this means, linearization around operating conditions, design (at each operating points) of a LTI controller, and interpolation of the LTI controllers in between operating conditions (often used in Aerospace and Automotive industries).
  **Pros**: Simplicity of design for a non linear system
  **Cons**: No *a priori* guarantee of stability nor robustness
- **But**: this differs from quasi-LPV representations where nonlinearities are hidden in some parameter descriptions (as seen later in the course)

LPV=LTV

- Theoretical analysis of LPV system properties (stability, controllability, observability), often falls into the framework of LTV systems or of nonlinear ones (for quasi-LPV representations), see (Blanchini).
What is a Linear Parameter Varying systems?

Some references

Those not to be ignored

- Modelling, identification: (Bruzelius, Bamieh, Lovera, Toth) + 2011 TCST Special Issue on "Applied LPV modelling and identification"
- Control (Shamma, Apkarian & Gahinet, Adams, Packard, Beker, Seiler, Grigoriadis ...)
- Stability, stabilization (Scherer, Wu, Blanchini ...)
- Geometric analysis (Bokor & Balas)
- Survey paper: Hoffmann & Werner, 2015
- Fault tolerant control: special issues by
  - Casavola, Rodrigues & Theilliol, 2015: in *International Journal of Robust and Nonlinear Control*

Some recent books

- R. Toth, Modeling and identification of linear parameter-varying systems, Springer 2010
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Different Models

According to the dependency on the parameter set, we may have several classes of models:

1. **Affine parameter dependency:**
   \[ A(\rho) = A_0 + A_1 \rho_1 + \ldots + A_N \rho_N \]

2. **Polynomial dependency:**
   \[ A(\rho) = A_0 + A_1 \rho + A_2 \rho^2 + \ldots + A_S \rho^S \]

3. **Rational dependency:**
   \[ A(\rho) = \left[ A_{n0} + A_{n1} \rho_{n1} + \ldots + A_{nN} \rho_{nN} \right] \left[ I + A_{d1} \rho_{d1} + \ldots + A_{dN} \rho_{dN} \right]^{-1} \]

**Brief insight in LFR Models**

Denoting the transfer matrix \( N(s) \) as:
\[
\begin{bmatrix}
  z_\Delta \\
  z
\end{bmatrix} = \begin{bmatrix}
  N_{11}(s) & N_{12}(s) \\
  N_{21}(s) & N_{22}(s)
\end{bmatrix} \begin{bmatrix}
  w_\Delta \\
  w
\end{bmatrix}
\]

The **Linear Fractional Representation (LFR)** gives then the transfer matrix from \( w \) to \( z \), and is referred to as the **upper Linear Fractional Transformation (LFT)**:

\[
F_u(N, \Delta) = N_{22} + N_{21} \Delta (I - N_{11} \Delta)^{-1} N_{12}
\]

This LFT exists and is well-posed if \((I - N_{11} \Delta)^{-1}\) is invertible.

**Figure:** System under LFT form

\( \Delta \) is defined such as: \( z_\Delta = \Delta(.)w_\Delta \)

It represents the parameter variations. \( \Delta(.) \) is a linear function of the parameter vector.

This will not be presented in the course. Please refer to (Apkarian & Gahinet; Scherer, Rantzer) for the use of LFT for robust analysis and design. This many need to study Integral Quadratic Constraints (IQCs).
Polytopic models

A polytopic system is represented as

$$\Sigma(\rho) = \sum_{k=1}^{Z} \alpha_k(\rho) \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}, \quad \text{with } \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

where $$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$ are LTI systems.

This representation is often used to rewrite an affine LPV system. Indeed, assuming that the parameters are bounded ($$\rho_i \in [\underline{\rho}_i, \bar{\rho}_i]$$), the vector of parameters evolves inside a polytope represented by $$Z = 2^N$$ vertices $$\omega_i$$, as

$$\rho \in \text{Co}\{\omega_1, \ldots, \omega_Z\}$$

(3)

It is then written as the convex combination:

$$\rho = \sum_{i=1}^{Z} \alpha_i \omega_i, \quad \alpha_i \geq 0, \quad \sum_{i=1}^{Z} \alpha_i = 1$$

(4)

where the vertices are defined by a vector $$\omega_i = [v_{i1}, \ldots, v_{iN}]$$ where $$v_{ij}$$ equals $$\rho_j$$ or $$\bar{\rho}_j$$.

The LTI system $$\begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix}$$ here corresponds to the LPV system frozen at the vertex $$k$$. 
From a generic affine LPV systems to a polytopic model

For a LPV system with 2 parameters, bounded \( [\underline{\rho}_{1,2}, \bar{\rho}_{1,2}] \), the corresponding polytope owns 4 vertices as:

\[
\mathcal{P}_\rho = \left\{ (\rho_1, \rho_2), (\rho_1, \bar{\rho}_2), (\underline{\rho}_1, \rho_2), (\underline{\rho}_1, \bar{\rho}_2) \right\}
\]  

(5)

The polytopic coordinates are \((\alpha_i)\) are obtained as:

\[
\begin{align*}
\omega_1 &= (\rho_1, \rho_2), \quad \alpha_1 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_1} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right) \\
\omega_2 &= (\rho_1, \bar{\rho}_2), \quad \alpha_2 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_1} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right) \\
\omega_3 &= (\underline{\rho}_1, \rho_2), \quad \alpha_3 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_1} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right) \\
\omega_4 &= (\underline{\rho}_1, \bar{\rho}_2), \quad \alpha_4 = \left( \frac{\rho_1 - \rho_1}{\bar{\rho}_1 - \rho_1} \right) \times \left( \frac{\rho_2 - \rho_2}{\rho_2 - \rho_2} \right)
\end{align*}
\]

(6)

where \(\rho_1\) and \(\rho_2\) are the instantaneous values of the parameters \((\rho_i^{(k)})\) in the implementation step.

The LPV system is then rewritten under the polytopic representation:

\[
\begin{pmatrix}
A(\rho_{1,2}) & B(\rho_{1,2}) \\
C(\rho_{1,2}) & D(\rho_{1,2})
\end{pmatrix} = \alpha_1 \begin{pmatrix}
A(\omega_1) & B(\omega_1) \\
C(\omega_1) & D(\omega_1)
\end{pmatrix} + \alpha_2 \begin{pmatrix}
A(\omega_2) & B(\omega_2) \\
C(\omega_2) & D(\omega_2)
\end{pmatrix} + \alpha_3 \begin{pmatrix}
A(\omega_3) & B(\omega_3) \\
C(\omega_3) & D(\omega_3)
\end{pmatrix} + \alpha_4 \begin{pmatrix}
A(\omega_4) & B(\omega_4) \\
C(\omega_4) & D(\omega_4)
\end{pmatrix}
\]

(7)
See (Boyd et al, 1994).
Let consider the nonlinear system

$$\Sigma_{NL} : \begin{align*}
\dot{x} &= f(x(t), w(t)) \\
z &= g(x(t), w(t))
\end{align*}$$

(8)

Suppose that, for each $x$, $w$ and $t$, there is a matrix $G(x, w, t) \in \Omega$ s.t.:

$$\begin{bmatrix} f(x, w) \\ g(x, w) \end{bmatrix} = G(x, w, t) \begin{bmatrix} x \\ w \end{bmatrix}$$

(9)

where $\Omega \in \mathbb{R}^{(n_x+n_z) \times (n_x+n_u)}$.

As said in (Boyd et al, 1994):
"Then of course every trajectory of the nonlinear system (8) is also a trajectory of the LDI defined by (9). If we can prove that every trajectory of the LDI defined by (9) has some property (e.g., converges to zero), then a fortiori we have proved that every trajectory of the nonlinear system (8) has this property."
Quarter vehicle dynamics

\[
\begin{align*}
    m_s \ddot{z}_s &= -k_s z_{def} - F_{\text{damper}} \\
    m_{us} \ddot{z}_{us} &= k_s z_{def} + F_{\text{damper}} - k_t (z_{us} - z_r)
\end{align*}
\]  \hfill (10)

\(z_{def} = z_s - z_{us}\) : damper deflection, \(\dot{z}_{def} = \dot{z}_s - \dot{z}_{us}\) : deflection velocity.

- The damper's characteristics: Force-Deflection-Deflection Velocity relation

\[
F_{\text{damper}} = g(z_{def}, \dot{z}_{def})
\]  \hfill (11)

where \(g\) can be linear or nonlinear.
LPV modelling of a quarter car vehicle suspension model (cont.)

A Semi-active nonlinear MR damper model [Gu et al., 2006, Nino-Juarez et al., 2008]

\[ F_{\text{damper}} = c_0 \dot{z}_{\text{def}} + k_0 z_{\text{def}} + f_I \tanh (c_1 \dot{z}_{\text{def}} + k_1 z_{\text{def}}) \]  

(12)

- The \( \tanh \) function allows to model the bi-viscous behavior.
- \( 5c_0, k_0, c_1, k_1 \): constant parameters. \( k_0, k_1 \) dedicated to the hysteresis behavior.
- \( f_I \) is a controllable force and depends on input current \( I \).

LPV model

Choosing \( \rho = \tanh (c_1 \dot{z}_{\text{def}} + k_1 z_{\text{def}}) \), and denoting \( u = f_I \) the control input, the quarter car model can be represented as:

\[
\begin{cases}
\dot{x}(t) = Ax(t) + B(\rho)u(t), \\
y(t) = Cx(t) + Du(t)
\end{cases}
\]

(13)
A brief insight in identification of LPV models

Global approaches - input/output models

\[ y(k) = -\sum_{i=1}^{na} a_i(\rho(k))y(k-i) + \sum_{j=1}^{nb} b_j(\rho(k))u(k-j) \]

(Bamieh & Giarre 99, 02): characterisation of persistency of excitation conditions for input-output LPV models

Previdi & Lovera 03, 04): NLPV model class (LFT feeded by a neural network model for the scheduling policy)

(Toth, 07 + book 2010): An LPV system can be viewed as a collection of "local" behaviours (associated with constant parameter values)

Global approaches - state space models

(Lee & Poolla): maximum likelihood (ML) algorithm for the identification of MIMO LPV-LFT models (PEM algorithm)

(Verhaegen et al, 02, 07, 09...): Supspace methods

Local approaches

Interpolation of locally identified LTI models... need to pay attention to:
- Input/output form (Toth, 07 + book 2010): interpolating transfer function coefficients
- State space form (Steinbuch et al, 03): consistency of state space basis
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Let consider the LPV syetsm

$$
\Sigma_{\rho} \left\{ \begin{array}{l}
\dot{x}(t) = A(\rho(t))x(t) + B(\rho(t))u(t), \quad x(0) = x_0 \\
y(t) = C(\rho(t))x(t) + D(\rho(t))u(t)
\end{array} \right. 
$$

What kind of properties we should pay attention to?

When $\rho$ is fixed (constant) the previous system is LTI and

- controllability, observability, stability, are uniquely defined
- controllability $\iff$ reachability, observability $\iff$ reconstructibility
- these properties are equivalent by a state change of basis.

But when $\rho(t)$ is time varying .....  

- these facts may not be true (asymptotic and exponential stability may differ)
- need to study properies of Linear Time-Varying systems.
- A generalization of the $\exp(At)$ is needed, defining the state transition matrix $\Phi(t,t_0,\rho(t))$
- For a change of basis $T(t)$ with $x(t) = T(t)x_{new}(t)$ then, $\dot{x}(t) = \dot{T}(t)x_{new}(t) + T(t)\dot{x}_{new}(t)$
Illustration for observability

In an analogous way the unobservability property is defined as: a state \( x(t) \) is not observable if the corresponding output vanishes, i.e. if the following holds: \( y(t) = \dot{y}(t) = \ddot{y}(t) = \ldots = 0 \). In the case of LTV systems it corresponds to:

Definition

The LPV system (14) is completely observable if \( \text{rank} \Theta = n \ \forall t \), where

\[
\Theta = \begin{bmatrix}
o_1^T & o_2^T & \ldots & o_n^T
\end{bmatrix}^T
\]

where \( o_1 = C(\rho) \) and \( o_{i+1} = o_i A + \dot{o}_i, \ i > 1 \) (for instance \( o_2 = \dot{\rho} \frac{\partial C(\rho)}{\partial \rho} + C(\rho) A(\rho) \)).

A weaker notion of observability can be defined for the LPV systems (14) in the functional sense \( \Theta \) function of \( \rho(t) \).

Definition

The LPV system (14) is structurally observable if \( \text{rank} \Theta = n \)

This does not guarantee that \( \Theta \) is invertible \( \forall t \) and for all parameter values.

Finally the above notion differ from the direct extension of the observability matrix for LTI systems, i.e

\[
\Theta = \begin{bmatrix}
C(\rho) \\
C(\rho) A(\rho) \\
\vdots \\
C(\rho) A^{n-1}(\rho)
\end{bmatrix}
\]

This definition is ONLY valid if \( \rho \) is constant, i.e. it corresponds to the observability of the LTI systems frozen at the values of the constant parameter vector \( \rho \).
Example

\[
\begin{align*}
\Sigma_1(\rho): \quad & \begin{cases}
\dot{x}(t) = A(\rho)x(t) \\
y(t) = C(\rho)x(t)
\end{cases} \\
\text{with} \\
A = \begin{pmatrix} 1 & 1 \\ \rho(t) & 2 \end{pmatrix}, \\ C = \begin{pmatrix} \rho(t) \\ 1 \end{pmatrix}
\end{align*}
\]

Observability matrix with \(\rho_f\) is a frozen value of \(\rho(t)\):

\[
\begin{pmatrix} \rho_f & 1 \\ 2\rho_f & \rho_f + 2 \end{pmatrix}
\]

which is of rank 2 apart for \(\rho_f = 0\). Therefore the LTI frozen systems are observable.

However the observability matrix of the considered time-varying system is given by:

\[
\begin{pmatrix} \rho(t) & 1 \\ \dot{\rho}(t) + 2\rho(t) & \rho(t) + 2 \end{pmatrix}
\]

which is of rank 2 in the functional sense. Therefore the structural rank of \(\Sigma_1(\rho)\) is 2. However it is of rank 1 if \(\rho\) satisfies \(\dot{\rho}(t) = \rho(t)^2\). The system is then not completely observable.

Therefore, for some specific parameter definitions, the parameter variations may therefore induce a loss of observability.
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Problem statement and facts

Recall

For LTI systems all notions of stability are equivalent: global/local, asymptotic/exponential, time-domain (Lyapunov)/frequency-domain (Bode, poles...).

Why stability analysis for LPV systems is not an easy task?

Let consider $\dot{x} = A(\rho(t))x$. Stability analysis is more involved (as for LTV systems) since:

- there is a set of solutions for a given $x_0$ (family of systems from $\rho$ variations)
- the system may be stable for frozen parameter values and unstable for varying parameters (as for switching systems)
- asymptotic and exponential stability are no more equivalent and cannot be characterized by the eigenvalues of $A(\rho(t))$.
- In term of design, we will often rely on the notion of quadratic stability (using quadratic Lyapunov function $V(x) = x^T PX$) which is stronger but easier to check for stability and simpler to use for control and observer design, see (Wu, PhD 95)

Robust or LPV? (Blanchini, 00 & 07)

- Robust analysis and control: dedicated to LTI systems subject to time-varying uncertainties
- LPV (or gain-scheduling) analysis and control: dedicated to LTV systems or to linearizations of non linear systems along the trajectory of $\rho$
Recall: robust stability with time-invariant uncertainties

This concept is very useful for the stability analysis of uncertain systems. Let us consider an uncertain system

\[ \dot{x} = A(\delta)x \]

where \( \delta \) is a parameter vector that belongs to an uncertainty set \( \Delta \).

**Problem statement**

Is the system asymptotically stable for all \( \delta \) in \( \Delta \)?

**Definition**

The considered system is said to be quadratically stable for all uncertainties \( \delta \in \Delta \) if there exists a (single) Lyapunov function \( V(x) = x^T P x \) with \( P = P^T > 0 \) s.t.

\[ A(\delta)^T P + PA(\delta) < 0, \text{ for all } \delta \in \Delta \] (15)

**Computation**

For polytopic uncertainties (convex set), i.e. if \( \rho \in \text{Co}\{\omega_1, \ldots, \omega_Z\} \), then, the problem becomes feasible since it remains to find \( P = P^T > 0 \) such that:

\[ A(\omega_i)^T P + PA(\omega_i) < 0, \text{ for } i = 1, \ldots, Z \]
Quadratic stability for time-varying parameters

Let us consider the LPV system

\[ \dot{x} = A(\rho(t))x \]

where \( \rho(t) \) is a time-varying parameter vector that belongs to an uncertainty set \( \Omega \).

Use of a single Lyapunov function

If there exists \( P = P^T > 0 \) such that:

\[ A(\rho(t))^T P + PA(\rho(t)) < 0, \forall \rho(t) \in \Omega \]

then the system is stable for arbitrarily fast time-varying uncertainties.

Remarks

- Quadratic stability implies exponential stability (Wu, 95)
- It is an infinite dimension problem (can be relaxed for polytopic uncertainties)
- It could be conservative since stability is checked for any variation of the parameters!

Pay attention in what follows: LPV system means TIME-VARYING parameters so a polytopic LPV system is not an uncertain polytopic system (in the latter case the coefficient \( \alpha_i \) of the polytopic description are constant even if unknown).
Parameter Varying Lyapunov functions

Let consider now a parameter dependent Lyapunov function $V_\rho(x(t)) = x(t)^T P(\rho)x(t) > 0$ for every $x \neq 0$ and $V(0) = 0$.

Uncertain systems ($\rho$ is time-invariant)

The uncertain system $\dot{x} = A(\rho)x$ is exponentially stable if there exists $V_\rho$ such that (classical approach for polytopic uncertain systems):

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) < 0, \forall \rho \in \mathbb{R}$$

LPV systems ($\rho$ is time-varying)

The uncertain system $\dot{x} = A(\rho(t))x$ is exponentially stable if there exists $V_\rho$ such that:

$$A(\rho)^T P(\rho) + P(\rho)A(\rho) + \sum_{i=1}^{N} \dot{\rho}_i \frac{\partial P(\rho)}{\partial \rho_i} < 0 \forall \rho(t) \in \mathbb{R}$$

which, in addition to bounded parameters, needs to consider rate-bounded parameter variations. Such a condition is more complex since:

- It involves the partial differentiation of $P$
- it has to be checked for all $\rho(t) \in \mathbb{R}$
- It implies to choose a parametrization of $P(\rho)$: from affine to polynomial
**Definition**

Given a parametrically dependent stable LPV system $\Sigma_\rho = (A(\rho), B(\rho), C(\rho), D(\rho))$ for zero initial conditions $x_0$. The induced $L_2$ norm is defined as:

$$||\Sigma_\rho||_{i, 2} = \sup_{\rho(t) \in \Omega} \sup_{w(t) \neq 0 \in L_2} \frac{||y||_2}{||u||_2}$$

which is often referred to as (by abuse of language) the $H_\infty$ gain $||\Sigma_\rho||_\infty$ of the LPV system.

**Theorem**

A sufficient condition for the $L_2$ stability of system $\Sigma_\rho$ is the generalized BRL, using parameter dependent Lyapunov functions, i.e assuming $|\dot{\rho}_i| < v_i, \forall i$, if there exists $P(\rho) > 0, \forall \rho$ s.t

$$\begin{bmatrix}
A(\rho)^T P(\rho) + P(\rho) A(\rho) + \sum_{i=1}^{N} v_i \frac{\partial P(\rho)}{\partial \rho_i}
& P(\rho) B(\rho)
& C(\rho)^T

B(\rho)^T P(\rho)
& -\gamma I
& D(\rho)^T

C(\rho)
& D(\rho)
& -\gamma I
\end{bmatrix} < 0, \forall i. \quad (16)$$

then $||\Sigma_\rho||_{i, 2} \leq \gamma$
Outline

1. What is a Linear Parameter Varying systems?

2. Modelling and identification of LPV systems

3. Some properties of LPV systems

4. Stability of LPV systems

5. LPV Control & Observation
   - The Dynamic Output feedback case
   - LPV observer design

6. Summary of LPV approach interests
Towards LPV control

The "gain scheduling" approach

Some references

- Modelling, identification: (Bruzelius, Bamieh, Lovera, Toth)
- Control (Shamma, Apkarian & Gahinet, Adams, Packard, Beker ...)
- Stability, stabilization (Scherer, Wu, Blanchini ...)
- Geometric analysis (Bokor & Balas)
The $H_\infty$/$LPV$ control problem

Definition

Find a LPV controller $C(\rho)$ s.t the closed-loop system is stable and for $\gamma_\infty > 0$, $\sup \frac{\|z\|^2}{\|w\|^2} < \gamma_\infty$,

- Unbounded set of LMIs (Linear Matrix Inequalities) to be solved ($\rho \in \Omega$)

A solution: The "polytopic" approach [C. Scherer et al. 1997]

- Problem solved off line for each vertex of a polytope (convex optimisation) (using here a single Lyapunov function i.e. quadratic stabilization).
- On-line the controller is computed as the convex combination of local linear controllers

$$C(\rho) = \sum_{k=1}^{2N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix}, \sum_{k=1}^{2N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0$$

- Easy implementation !!
The $H_\infty/LPV$ control problem

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Find a LPV controller $C(\rho)$ s.t the closed-loop system is stable and for $\gamma_\infty > 0$, $\sup \|z\|^2_2 < \gamma_\infty$,

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- Easy implementation!!
LPV control design

Dynamical LPV generalized plant:

$$\Sigma(\rho) : \begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(\rho) & B_1(\rho) & B_2(\rho) \\ C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\ C_2(\rho) & D_{21}(\rho) & D_{22}(\rho) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}$$  \hspace{1cm} (17)

LPV controller structure:

$$S(\rho) : \begin{bmatrix} \dot{x}_c \\ u \end{bmatrix} = \begin{bmatrix} A_c(\rho) & B_c(\rho) \\ C_c(\rho) & D_c(\rho) \end{bmatrix} \begin{bmatrix} x_c \\ y \end{bmatrix}$$  \hspace{1cm} (18)

LPV closed-loop system:

$$CL(\rho) : \begin{bmatrix} \dot{z} \\ z \end{bmatrix} = \begin{bmatrix} A(\rho) & B(\rho) \\ C(\rho) & D(\rho) \end{bmatrix} \begin{bmatrix} \xi \\ w \end{bmatrix}$$  \hspace{1cm} (19)
LPV control design

Dynamical LPV generalized plant:

\[
\Sigma(\rho) : \begin{bmatrix}
\dot{x} \\
z \\
y
\end{bmatrix} = \begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
C_1(\rho) & D_{11}(\rho) & D_{12}(\rho) \\
C_2(\rho) & D_{21}(\rho) & D_{22}(\rho)
\end{bmatrix} \begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\]  

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y
\end{bmatrix}
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\[
CL(\rho) : \begin{bmatrix}
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\end{bmatrix} = \begin{bmatrix}
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\mathcal{C}(\rho) & \mathcal{D}(\rho)
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\xi \\
w
\end{bmatrix}
\]
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\dot{z} \\
y
\end{bmatrix} =
\begin{bmatrix}
A(\rho) & B_1(\rho) & B_2(\rho) \\
\frac{C_1(\rho)}{D_{11}(\rho)} & \frac{D_{12}(\rho)}{D_{21}(\rho)} \\
\frac{C_2(\rho)}{D_{22}(\rho)}
\end{bmatrix}
\begin{bmatrix}
x \\
w \\
u
\end{bmatrix}
\tag{17}
\]

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\mathcal{C}(\rho) & \mathcal{D}(\rho)
\end{bmatrix}
\begin{bmatrix}
\xi \\
w
\end{bmatrix}
\tag{19}
\]
The Dynamic Output feedback case

$H_\infty$ criteria Apkarian et al. [TAC, 1995]

Stabilize system $CL(\rho)$ (find $K > 0$) while minimizing $\gamma_\infty$.

$$
\begin{bmatrix}
A(\rho)^T K + K A(\rho) & K B_\infty(\rho) & C_\infty(\rho)^T \\
B_\infty(\rho)^T K & -\gamma_\infty^2 I & D_\infty(\rho)^T \\
C_\infty(\rho) & D_\infty(\rho) & -I
\end{bmatrix} < 0
$$

Infinite set of LMIs to solve ($\rho \in \Omega$) ($\Omega$ is convex)


LFT, Gridding, Polytopic
LPV control design

$H_\infty$ criteria Apkarian et al. [TAC, 1995]

Stabilize system $CL(\rho)$ (find $K > 0$) while minimizing $\gamma_\infty$.

$$
\begin{bmatrix}
A(\rho)^T K + K A(\rho) & KB_\infty(\rho) & C_\infty(\rho)^T \\
B_\infty(\rho)^T K & -\gamma_\infty^2 I & D_\infty(\rho)^T \\
C_\infty(\rho) & D_\infty(\rho) & -I
\end{bmatrix} < 0
$$

Infinite set of LMIs to solve ($\rho \in \Omega$) ($\Omega$ is convex)


LFT, Gridding, Polytopic
LPV control design

Polytopic approach

Solve the LMIs at each vertex of the polytope formed by the extremum values of each varying parameter, with a common $K$ Lyapunov function.

\[
C(\rho) = \sum_{k=1}^{2^N} \alpha_k(\rho) \begin{bmatrix}
A_c(\omega_k) & B_c(\omega_k) \\
C_c(\omega_k) & D_c(\omega_k)
\end{bmatrix}
\]

where,

\[
\alpha_k(\rho) = \frac{\prod_{j=1}^{N} |\rho_j - C_c(\omega_k)_j|}{\prod_{j=1}^{N} (\bar{\rho}_j - \rho_j)}
\]

where $C_c(\omega_k)_j = \{\bar{\rho}_j \text{ if } (\omega_k)_j = \rho_j \text{ or } \rho_j\}$ otherwise.

\[
\sum_{k=1}^{2^N} \alpha_k(\rho) = 1, \alpha_k(\rho) > 0
\]
Polytopic approach

Solve the LMIs at each vertex of the polytope formed by the extremum values of each varying parameter, with a common $K$ Lyapunov function.

$$C(\rho) = \sum_{k=1}^{2^N} \alpha_k(\rho) \begin{bmatrix} A_c(\omega_k) & B_c(\omega_k) \\ C_c(\omega_k) & D_c(\omega_k) \end{bmatrix} $$
LPV/\(\mathcal{H}_\infty\) control synthesis

Proposition - feasibility (brief) Scherer et al. (1997)

Solve the following problem at each vertices of the parametrized points (illustration with 2 parameters):

\[\gamma^* = \min \gamma \text{ s.t. (21)}\]

\[
\begin{pmatrix}
A \mathbf{X} + B_2 \mathbf{C}(\rho_1, \rho_2) + (\ast)^T \\
\bar{A}(\rho_1, \rho_2) + A^T \\
B_1^T \\
C_1 \mathbf{X} + D_{12} \mathbf{C}(\rho_1, \rho_2)
\end{pmatrix}
\begin{pmatrix}
\mathbf{X} \\
\mathbf{I} \\
\mathbf{Y}
\end{pmatrix}
\succ 0
\]

\[
\begin{pmatrix}
\mathbf{Y}A + \bar{B}(\rho_1, \rho_2)C_2 + (\ast)^T \\
B_1^T \mathbf{Y} + D_{21}^T \bar{B}(\rho_1, \rho_2)^T \\
\mathbf{C}_1 \\
\mathbf{D}_{11} \end{pmatrix}
\begin{pmatrix}
(\ast)^T \\
(\ast)^T \\
(\ast)^T \\
(\ast)^T
\end{pmatrix}

\begin{pmatrix}
(\ast)^T \\
(\ast)^T \\
(\ast)^T \\
-\gamma \mathbf{I}
\end{pmatrix}

\begin{pmatrix}
\mathbf{I} \\
\mathbf{Y}
\end{pmatrix}

\preceq 0
\]
LPV control & Observation

The Dynamic Output feedback case

LPV/\mathcal{H}_\infty\ control synthesis

Proposition - reconstruction (brief) Scherer et al. (1997)

Reconstruct the controllers as,

\begin{align*}
\text{solve} \quad (23) \quad & \begin{bmatrix} \rho_1, \rho_2 \\ \rho_1, \rho_2 \\ \rho_1, \rho_2 \\ \rho_1, \rho_2 \end{bmatrix} \\
\begin{cases} 
C_c(\rho_1, \rho_2) &= \tilde{C}(\rho_1, \rho_2)M^{-T} \\
B_c(\rho_1, \rho_2) &= N^{-1}\tilde{B}(\rho_1, \rho_2) \\
A_c(\rho_1, \rho_2) &= N^{-1}(\tilde{A}(\rho_1, \rho_2) - YAX - NB_c(\rho_1, \rho_2)C_2X) \\
&\quad - YB_2C_c(\rho_1, \rho_2)M^T \end{cases} \\
\end{align*}

(22)

(23)

where $M$ and $N$ are defined such that $MN^T = I - XY$ which may be chosen by applying a singular value decomposition and a Cholesky factorization.
Definition LPV observers

**Definition**

Let consider the LPV system:

\[
\begin{align*}
\dot{x}(t) &= A(\rho)x(t) + B(\rho)u(t) \\
y(t) &= C(\rho)x(t)
\end{align*}
\]  

(24)

The following LPV state space representation

\[
\hat{x}(t) = A(\rho)\hat{x}(t) + B(\rho)u(t) + L(\rho)(y(t) - C(\rho)\hat{x}(t))
\]

(25)

\[\hat{x}_0 \text{ to be defined}\]

is said to be an observer for (24) if

\[\lim_{t \to \infty} (\hat{x}(t) - x(t)) \to 0 \quad \forall \rho(t) \in \Omega\]

where \(\hat{x}(t) \in \mathbb{R}^n\) is the estimated state of \(x(t)\) and \(L(\rho)\) is the \(n \times p\) observer gain matrix to be designed.
Some issues for LPV observer design

The estimated error, $e(t) := x(t) - \hat{x}(t)$, satisfies:

$$\dot{e}(t) = (A - LC)(\rho)e(t)$$  \hspace{1cm} (26)

The two main problems to be handle are then

- What observability property shall we consider?
- What parameter dependency should we define for $L(\rho)$?

**Quadratic detectability (Wu, 95)**

A simple solution is to consider a single Lyapunov function in order to guarantee the quadratic detectability, i.e:

$$(A(\rho) - L(\rho)C(\rho))^T P + P(A(\rho) - L(\rho)C(\rho)) < 0$$

**Some remarks:**

- The previous problem can be solved using a polytopic approach only if $C(\rho) = C$, a constant matrix
- If this is not solvable, one can try using Parameter dependent Lyapunov functions, but the coupling between $L(\rho)$ and $P(\rho)$ will lead to solved non affine LMIs (a polynomial or a gridding approach is then needed).
Some issues for LPV observer design (2)

On key issue in observer implementation concerns the knowledge of $\rho(t)$. While previously the result is valid if $\rho(t)$ is perfectly known, such a following observer description must be used if $\rho(t)$ is estimated:

$$\dot{\hat{x}}(t) = A(\hat{\rho})\hat{x}(t) + B(\hat{\rho})u(t) + L(\hat{\rho})(y(t) - C(\hat{\rho})\hat{x}(t))$$ \hspace{1cm} (27)

Denoting $\Delta A = A(\rho) - A(\hat{\rho})$, $\Delta B = B(\rho) - B(\hat{\rho})$, $\Delta C = C(\rho) - C(\hat{\rho})$, and $\Delta L = L(\rho) - L(\hat{\rho})$, this leads for the estimation error equation:

$$\dot{e}(t) = (A - LC)(\hat{\rho})e(t) + (\Delta A + L(\hat{\rho}).\Delta C)x + \Delta Bu(t)$$ \hspace{1cm} (28)

If $C(\rho) = C$ and $B(\rho)$ are constant matrices, then we get the uncertain estimated error system

$$\dot{e}(t) = (A(\hat{\rho}) - L(\hat{\rho})C)e(t) + \Delta Ax(t)$$ \hspace{1cm} (29)

The stability analysis is indeed more involved due to the state vector $x$ (see (Daafouz et al, 2010) for the discrete-time case). Either $\Delta Ax(t)$ should be considered as a disturbance, or a state augmentation approach is to be used (which has to be done in closed-loop control).

Observer-based control

For control design in the latter case, the following state feedback should be used:

$$u(t) = -F(\hat{\rho})\hat{x}(t)$$
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6. Summary of LPV approach interests
Interest of the LPV approach

LPV is a key tool to the control of complex systems.

*Some examples:*

Modelling of complex systems (non linear)

- Use of a quasi-LPV representation to include non linearities in a linear state space model (even delays)
- Transformation of constraints (e.g. saturation) into an 'external' parameter
- Modelling of LTV, hybrid (e.g. switching control)

*BUT:*

A q-LPV system is not equivalent to the non linear one:

- **stability:** $\rho = \rho(x(t),t)$ is assumed to be bounded... so are the state trajectories
- **controllability:** some non controllable modes of a non linear system may vanish according to the LPV representation
- **observability:** unobservability may occur for some specific parameter variations
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Interest of the LPV approach

Some of works using LPV approaches - former PhD students

Gain-scheduled control

- Account for various operating conditions using a variable "equilibrium point": (Gauthier 2007)
- Control with real-time performance adaptation using parameter dependent weighting functions from endogenous or exogenous parameters (Poussot 2008, Do 2011)
- Control under computation constraints: $H_\infty$ variable sampling rate controller with sampling dependent performances (Robert 2007, Roche 2011, Robert et al., IEEE TCST 2010)

Coordination of several actuators for MIMO systems

- An LPV structure for control allocation Poussot et al. (CEP 2011)
- Selection of a specific parameter for the control activation (of each actuator) Poussot et al. (VSD 2011), Doumiati et al (EJC 2013), Fergani et al (IEEE TVT 2015)

Incorporate fault-(diagnosis, accommodation, tolerant control) properties

Some *Grenoble* PhD students on LPV control

- Manh Quan Nguyen, "LPV approaches for modelling and control of vehicle dynamics: application to a small car pilot plant with ER dampers", PhD GIPSA-lab, Univertisté Grenoble Alpes, 2016.
- Waleed Nwesaty, "LPV/$H_\infty$ control design of on-board energy management systems for electrical vehicles", PhD GIPSA-lab, Univertisté Grenoble Alpes, 2015.
- Soheib Fergani, "Robust LPV/$H_\infty$ MIMO control for vehicle dynamics, PhD GIPSA-lab, Univertisté Grenoble Alpes, 2014.
- Maria Rivas, "Modeling and Control of a Spark Ignited Engine for Euro 6 European Normative", PhD, GIPSA-lab / RENAULT, Grenoble INP, 2012.
- David Hernandez, "Robust control of hybrid electro-chemical generators", PhD, GIPSA-lab / G2Elab, Grenoble INP, 2011.
- Emilie Roche, "Commande Linéaire à Paramètres Variants discrète à échantillonnage variable : application à un sous-marin autonome", PhD, GIPSA-lab, Grenoble INP, 2011.
- Corentin Briat, "Robust control and observation of LPV time-delay systems", PhD, GIPSA-lab, INP Grenoble, 2008.
- Christophe Gauthier, "Commande multivariable de la pression d’injection dans un moteur Diesel Common Rail", PhD, LAG / DELPHI, Grenoble INP, 2007.
Some references


